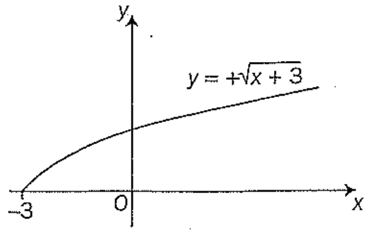
VJM's OMICRON "replaces omicron on VLE"

1 (a) Draw the graphs of $y = |x - 2|$ and $y = |x| + 1$ on the same axes.

(b) Using your graph or otherwise, solve the inequality
$$|x - 2| > |x| + 1$$

2 The graph of $f(x) = +\sqrt{x+3}, \quad x \geqslant -3$ is shown in this diagram.

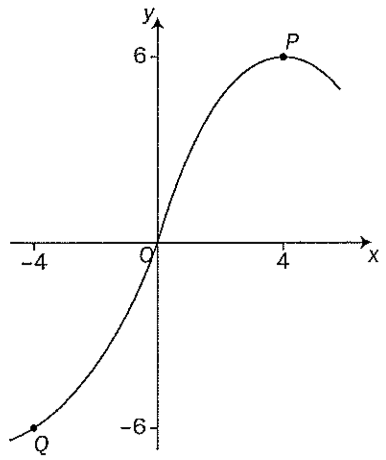


(a) Find an expression for the inverse function $f^{-1}(x)$.

(b) Find the values of (i) $ff^{-1}(2)$ (ii) $ff(1)$.

(c) Sketch the graph of $y = f^{-1}(x)$
Give the coordinates of points where the graph meets the axes.

3 The graph of the function $y = f(x)$ is shown in this diagram.
The curve passes through the points $P(4, 6)$, $Q(-4, -6)$ and the origin O.



On three separate diagrams, sketch the graphs with the equations

(a) $y = f(|x|)$ (b) $y = |f(x)|$ (c) $y = f\left(\frac{1}{2}x - 1\right)$

On each of your diagrams, give the coordinates of the images of points P and Q.

4 The functions f and g are defined by
$f: x \longrightarrow 3x^2 - 1, \quad x \in \mathbb{R},$
$g: x \longrightarrow \frac{2}{x-1}, \quad x \in \mathbb{R}, \ x \neq 1$

(a) Find the inverse function g^{-1}, stating its domain and range.

(b) Show that the composite function fg is
$$fg: x \longrightarrow \frac{11 + 2x - x^2}{(x-1)^2}$$

(c) Find the coordinates of the points where the graph $y = fg(x)$ cuts the x-axis, giving your answers correct to 2 decimal places.

(d) Find the equations of the two vertical asymptotes on the graph of the composite function $y = gf(x)$.

5 (a) Prove the identity $\dfrac{\sin\theta - \sin 2\theta}{\cos\theta - \cos 2\theta - 1} \equiv \tan\theta$

(b) Solve the equation $\dfrac{3\cos 2\theta}{\cos 2\theta + 1} = 2(\tan\theta + 1)$ for $0° < \theta < 360°$

(c) Write $\tan 2\theta$ in terms of $\tan\theta$.
Hence, find the exact value of $\tan 22\frac{1}{2}°$ as a surd.

6 The curve C has the equation $y = 5\cos 2x - 12\sin 2x$ where $x \geqslant 0$.

(a) Find the equation of the tangent to C at the point where $x = 0$.
Find the coordinates of the point A where the tangent cuts the x-axis.

(b) Express y in the form $R\cos(2x + \alpha)$
where $R > 0$ and α is an acute angle.
Give the value of α in radians correct to 4 significant figures.

(c) Point B is the point of intersection of the curve C and the x-axis which is closest to the origin.
Find the coordinates of B, correct to 3 significant figures.

(d) Find the distance AB.

7 A boiler operates at a high temperature. When it is switched off, its temperature T°C falls over a time of t minutes according to this equation:

$T = 20 + 800e^{-0.1t}, \quad t \geqslant 0.$

(a) Find the temperature of the boiler at the instant when it is switched off.

(b) How long does it take, to the nearest tenth of a minute, for the boiler's temperature to fall to 400 °C?

(c) What is the rate at which the temperature is falling 5 minutes after the boiler has been switched off? Give your answer in C degrees per minute to 3 significant figures.

(d) As t increases, what is the limiting value of T?

8 (a) Differentiate with respect to x

(i) x^4e^{2x} (ii) $\dfrac{3x^2}{\cos x}$ (iii) $\sin^3 x$

(b) Find $\dfrac{dy}{dx}$ in terms of y when

(i) $x = \sin^2 y$ (ii) $x = \sin(y^2)$ (iii) $x = a^y$

9 (a) Given that $f(x) = x^3 + x^2 - 3x - 9$, show that the equation $f(x) = 0$ has root in the interval $[2, 3]$.

(b) Show that $x_n = \sqrt{\dfrac{3x+9}{x+1}}$, $x_n \neq -1$ can be used as an iterative formula to solve the equation $f(x) = 0$.

(c) Taking $x_0 = 1$, find the values of x_1, x_2, x_3, x_4 and x_5 to 4 decimal places.
Hence, write down the solution of $f(x) = 0$ correct to 2 decimal places.

(d) Find a negative value of x_0 (other than $x = -1$) which does not give a valid value for x_1.
Give a reason for your choice.

10 (a) Express $\frac{x^2+2}{(2x-1)(x+1)^2}$ in partial fractions.

(b) Hence, prove that $\int_1^2 \frac{x^2+2}{(2x-1)(x+1)^2} = \frac{1}{6}(\ln 27 - 1)$

11 Given that $f(x) = \frac{4+x}{(1+x)(1-2x)}$

(a) express $f(x)$ in partial fractions

(b) hence, evaluate $\int_0^{\frac{1}{4}} f(x)\,dx$, giving your answer in the form $\ln(k\sqrt{2})$

(c) expand $f(x)$ using the binomial theorem up to and including the term in x^3. Simplify each term as far as possible. State the range of values of x for which the expansion is valid.

12 The curve C has the equation $4x^2 - 3xy + y^2 = 4$

(a) Find the gradient of C at the point $(1,3)$.

(b) Find the equation of the tangent to C at the point $(1,3)$.

13 (a) Using the substitution $u^2 = e^x - 1$ show that the integral $\int \frac{e^{2x}}{\sqrt{e^x-1}}\,dx$ is transformed to $2\int (u^2+1)\,du$

(b) Show that $\int_0^1 \frac{e^{2x}}{\sqrt{e^x-1}}\,dx = \frac{2}{3}(e+2)\sqrt{e-1}$

14 Find

(a) $\int xe^{2x}\,dx$

(b) $\int xe^{2x^2}\,dx$

ANSWERS

1 a) sketch b) $x < \frac{1}{2}$

2. a) $f^{-1}(x) = x^2 - 3$. c) sketch
b) (i) $f(1) = 2$
(ii) $f(2) = \sqrt{5}$

3. sketches

4. a) $g^{-1}(x) = \frac{2}{x} + 1$ $x \in \mathbb{R}, x \neq 0$ $g^{-1}(x) \in \mathbb{R}, g^{-1}(x) \neq 1$

b) proof
c) $(-2.464, 0)$ $(4.464, 0)$

d) $x = \pm\sqrt{\frac{2}{3}}$

5. a) proof b) $\theta = 161.6°, 341.6°, 135°, 315°$
c) $-1 + \sqrt{2}$

6. a) $24x + y = 5$ A is $\left(\frac{5}{24}, 0\right)$

b) $13\cos(2x + 1.176)$
c) B $(0.197, 0)$
d) 0.011 units

7. a) 820°C b) 7.4 mins c) −48.5°C/min
d) $T \to 20°C$

8. a) (i) $2x^3e^{2x}(2+x)$ (ii) $\frac{6x\cos x + 3x^2 \sin x}{\cos^2 x}$.

(iii) $3\cos x \sin^2 x$.

b) (i) $\dfrac{1}{2\sin y\cos y}$ (ii) $\dfrac{1}{2y\cos(y^2)}$ (iii) $\dfrac{1}{a^y \ln a}$.

9. a) change of sign b) proof

c) 2.4495, 2.1770, 2.2110, 2.2065
2.2071
so $x = 2.21$ (2dp)

d) $x_0 = -2$ gives $x_1 = \sqrt{-3}$
which cannot be evaluated

10. a) $\dfrac{1}{2x-1} - \dfrac{1}{(x+1)^2}$ $(B = 0)$

b) proof.

11. a) $\dfrac{1}{1+x} + \dfrac{3}{1-2x}$ b) $k = \dfrac{5}{2}$

c) $4 + 5x + 13x^2 + 23x^3$ $-\dfrac{1}{2} < x < \dfrac{1}{2}$

12 a) $\dfrac{1}{3}$ b) $x - 3y + 8 = 0$

13 Proof.

14. a) $\dfrac{1}{2}xe^{2x} - \dfrac{1}{4}e^{2x} + c$ b) $\dfrac{1}{4}e^{2x^2} + c$

~~15~~.