Question 13 (***)

The curve C has equation

$$2\cos 3x \sin y = 1, \quad 0 \le x, y \le \pi .$$

a) Show that

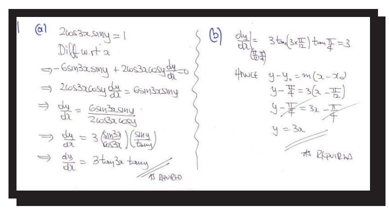
$$\frac{dy}{dx} = 3\tan 3x \tan y .$$

The point $P\left(\frac{\pi}{12}, \frac{\pi}{4}\right)$ lies on C.

b) Show that an equation of the tangent to C at P is

$$y = 3x .$$

$\boxed{\text{proof}}$



(a) $2\cos 3x \sin y = 1$

Diff w.r.t x

$\Rightarrow -6\sin 3x \sin y + 2\cos 3x \cos y \frac{dy}{dx} = 0$

$\Rightarrow 2\cos 3x \cos y \frac{dy}{dx} = 6\sin 3x \sin y$

$\Rightarrow \frac{dy}{dx} = \frac{6\sin 3x \sin y}{2\cos 3x \cos y}$

$\Rightarrow \frac{dy}{dx} = 3\left(\frac{\sin 3x}{\cos 3x}\right)\left(\frac{\sin y}{\cos y}\right)$

$\Rightarrow \frac{dy}{dx} = 3\tan 3x \tan y$

AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{12}, \frac{\pi}{4}\right)} = 3\tan\left(3\times\frac{\pi}{12}\right)\tan\frac{\pi}{4} = 3$

Hence $y - y_0 = m(x - x_0)$

$y - \frac{\pi}{4} = 3\left(x - \frac{\pi}{12}\right)$

$y - \cancel{\frac{\pi}{4}} = 3x - \cancel{\frac{\pi}{4}}$

$y = 3x$

AS REQUIRED

Question 14 (***)

A curve has equation

$$3x^2 - xy + y^2 + 2x - 4y = 1.$$

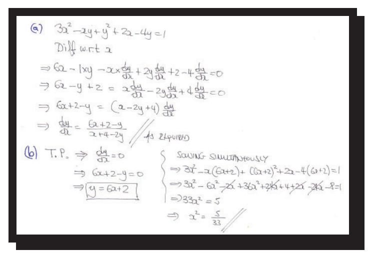
a) Show clearly that

$$\frac{dy}{dx} = \frac{2 + 6x - y}{4 - 2y + x}.$$

b) Hence show further that the value of x at the stationary points of the curve satisfies the equation

$$x^2 = \frac{5}{33}.$$

$\boxed{\text{proof}}$



(a) $3x^2 - xy + y^2 + 2x - 4y = 1$

Diff w.r.t x

$\Rightarrow 6x - 1xy - x \times \frac{dy}{dx} + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$

$\Rightarrow 6x - y + 2 = x\frac{dy}{dx} - 2y\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = (x - 2y + 4)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{6x+2-y}{x+4-2y}$ // AS REQUIRED

(b) T.P. $\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 6x + 2 - y = 0$

$\Rightarrow \boxed{y = 6x + 2}$

SOLVING SIMULTANEOUSLY

$\Rightarrow 3x^2 - x(6x+2) + (6x+2)^2 + 2x - 4(6x+2) = 1$

$\Rightarrow 3x^2 - 6x^2 - \cancel{2x} + 36x^2 + \cancel{24x} + 4 + \cancel{2x} - \cancel{24x} - 8 = 1$

$\Rightarrow 33x^2 = 5$

$\Rightarrow x^2 = \frac{5}{33}$ //

Question 39 **(****)**

The equation of a curve is given implicitly by

$$4y + y^2 e^{3x} = x^3 + C ,$$

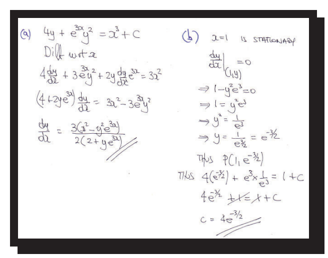
where C is a non zero constant.

a) Find a simplified expression for $\frac{dy}{dx}$.

The point $P(1,k)$, where $k > 0$, is a stationary point of the curve.

b) Find an exact value for C.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{3\left(x^2 - y^2 e^{3x}\right)}{2\left(2 + y e^{3x}\right)}}$, $\boxed{C = 4e^{-\frac{3}{2}}}$


(a) $4y + e^{3x}y^2 = x^3 + C$

Diff w.r.t x

$4\frac{dy}{dx} + 3e^{3x}y^2 + 2y\frac{dy}{dx}e^{3x} = 3x^2$

$(4+2ye^{3x})\frac{dy}{dx} = 3x^2 - 3e^{3x}y^2$

$\frac{dy}{dx} = \frac{3(x^2 - y^2e^{3x})}{2(2+ye^{3x})}$

(b) $x=1$ IS STATIONARY

$\left.\frac{dy}{dx}\right|_{(1,y)} = 0$

$\Rightarrow 1 - y^2e^3 = 0$

$\Rightarrow 1 = y^2e^3$

$\Rightarrow y^2 = \frac{1}{e^3}$

$\Rightarrow y = \frac{1}{e^{3/2}} = e^{-3/2}$

Thus $P(1, e^{-3/2})$

Thus $4(e^{-3/2}) + e^3 \times \frac{1}{e^3} = 1 + C$

$4e^{-3/2} \cancel{+1} = \cancel{1} + C$

$C = 4e^{-3/2}$

Question 40 (****)

A curve C has implicit equation

$$y = \frac{2x+1}{xy+3}.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of x and y.

b) Show that there is **no** point on C, where the tangent is parallel to the y axis.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{2-y^2}{2xy+3}}$

(a) $y = \frac{2x+1}{xy+3}$

$\Rightarrow xy^2 + 3y = 2x+1$

• Diff w.r.t x

$\Rightarrow y^2 + x\left(2y\frac{dy}{dx}\right) + 3\frac{dy}{dx} = 2$

$\Rightarrow (2xy+3)\frac{dy}{dx} = 2 - y^2$

$\Rightarrow \frac{dy}{dx} = \frac{2-y^2}{2xy+3}$

(b) SUPPOSE WE COULD GET SUCH TANGENT

IE INFINITE GRADIENT

IE DENOMINATOR ZERO

IE $\boxed{2xy+3=0}$

SOLVE SIMULTANEOUSLY WITH CURVE

$2xy = -3$

$\boxed{y = -\frac{3}{2x}}$ SUB INTO CURVE

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{x\left(\frac{-3}{2x}\right)+3}$

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{-\frac{3}{2}+3}$

$\Rightarrow -\frac{3}{2x} = \frac{2x+1}{\frac{3}{2}}$

$\Rightarrow -\frac{3}{2x} = \frac{4x+2}{3}$

$\Rightarrow -9 = 8x^2 + 4x$

$\Rightarrow 0 = 8x^2 + 4x + 9$

$b^2 - 4ac$

$= 16 - 4 \times 8 \times 9$

$= 16 - 288 < 0$

$\therefore$ NO SOLUTIONS

$\therefore$ NO VERTICAL TANGENT

Question 49 (****)

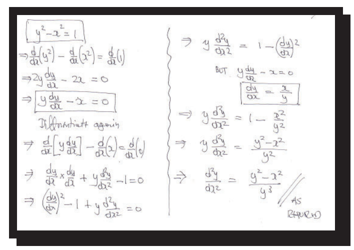
The equation of a curve is given implicitly by

$$y^2 - x^2 = 1, \quad |y| \geq 1 \ .$$

Show clearly that

$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3} .$$

$\boxed{\text{proof}}$



$\boxed{y^2 - x^2 = 1}$

$\Rightarrow \frac{d}{dx}(y^2) - \frac{d}{dx}(x^2) = \frac{d}{dx}(1)$

$\Rightarrow 2y\frac{dy}{dx} - 2x = 0$

$\Rightarrow \boxed{y\frac{dy}{dx} - x = 0}$

Differentiate again

$\Rightarrow \frac{d}{dx}\left[y\frac{dy}{dx}\right] - \frac{d}{dx}(x) = \frac{d}{dx}(0)$

$\Rightarrow \frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} - 1 = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 - 1 + y\frac{d^2y}{dx^2} = 0$

$\Rightarrow y\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$

BUT $y\frac{dy}{dx} - x = 0$

$\boxed{\frac{dy}{dx} = \frac{x}{y}}$

$\Rightarrow y\frac{d^2y}{dx^2} = 1 - \frac{x^2}{y^2}$

$\Rightarrow y\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$ // AS REQUIRED

Question 4 (**+)

The binomial expression $(1+x)^{\frac{1}{3}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

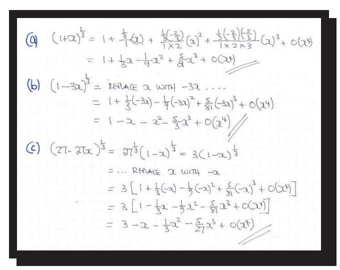
a) Determine the expansion of $(1+x)^{\frac{1}{3}}$, up and including the term in x^3.

b) Use the expansion of part **(a)** to find the expansion of $(1-3x)^{\frac{1}{3}}$, up and including the term in x^3.

c) Use the expansion of part **(a)** to find the expansion of $(27-27x)^{\frac{1}{3}}$, up and including the term in x^3.

$$\boxed{1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+O\left(x^4\right)}, \quad \boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)},$$

$$\boxed{3-x-\frac{1}{3}x^2-\frac{5}{27}x^3+O\left(x^4\right)}$$



(a) $(1+x)^{\frac{1}{3}} = 1+\frac{\frac{1}{3}}{1}(x)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times2}(x)^2+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\times2\times3}(x)^3+O(x^4)$

$= 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+O(x^4)$

(b) $(1-3x)^{\frac{1}{3}} =$ REPLACE x WITH $-3x$ ....

$= 1+\frac{1}{3}(-3x)-\frac{1}{9}(-3x)^2+\frac{5}{81}(-3x)^3+O(x^4)$

$= 1-x-x^2-\frac{5}{3}x^3+O(x^4)$

(c) $(27-27x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1-x)^{\frac{1}{3}} = 3(1-x)^{\frac{1}{3}}$

$= \ldots$ REPLACE x WITH $-x$

$= 3\left[1+\frac{1}{3}(-x)-\frac{1}{9}(-x)^2+\frac{5}{81}(-x)^3+O(x^4)\right]$

$= 3\left[1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3+O(x^4)\right]$

$= 3-x-\frac{1}{3}x^2-\frac{5}{27}x^3+O(x^4)$

Question 5 (**+)

$$f(x) = \frac{5x+3}{(1-x)(1+3x)}, \quad |x| < \frac{1}{3}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find the series expansion of $f(x)$, up and including the term in x^3.

$\boxed{}$, $\boxed{f(x) = \frac{2}{1-x} + \frac{1}{1+3x}}$, $\boxed{f(x) = 3 - x + 11x^2 - 25x^3 + O(x^4)}$

(a) $f(x) = \frac{5x+3}{(1-x)(1+3x)} = \frac{A}{1-x} + \frac{B}{1+3x}$

$\boxed{5x+3 \equiv A(1+3x) + B(1-x)}$

If $x=1 \Rightarrow 8 = 4A \Rightarrow A=2$

If $x=-\frac{1}{3} \Rightarrow \frac{4}{3} = \frac{4}{3}B \Rightarrow B=1$

$f(x) = \frac{2}{1-x} + \frac{1}{1+3x}$ //

(b) $f(x) = 2(1-x)^{-1} + (1+3x)^{-1}$

- $2(1-x)^{-1} = 2\left[1 + \frac{-1}{1}(-x) + \frac{(-1)(-2)}{1\times 2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-x)^3 + O(x^4)\right]$
  $= 2\left[1 + x + x^2 + x^3 + O(x^4)\right]$
  $= 2 + 2x + 2x^2 + 2x^3 + O(x^4)$
- $(1+3x)^{-1} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + O(x^4)$
  $= 1 - 3x + 9x^2 - 27x^3 + O(x^4)$

REPLACED x BY $-3x$ IN PREVIOUS EXPANSION OR FIND IT FROM SCRATCH

ADD $f(x) = 3 - x + 11x^2 - 25x^3 + O(x^4)$

Question 6 (**+)

$$f(x) = \frac{2x}{(1+2x)^3}, \quad x \neq -\frac{1}{2}.$$

a) Find the first 4 terms in the series expansion of $f(x)$.

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\boxed{}$, $\boxed{f(x) = 2x - 12x^2 + 48x^3 - 160x^4 + O(x^5)}$, $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$

(a) $f(x) = \frac{2x}{(1+2x)^3} = 2x(1+2x)^{-3}$

$= 2x\left[1 + \frac{-3}{1}(2x)^1 + \frac{(-3)(-4)}{1\times 2}(2x)^2 + \frac{(-3)(-4)(-5)}{1\times 2\times 3}(2x)^3 + O(x^4)\right]$

$= 2x\left[1 - 6x + 24x^2 - 80x^3 + O(x^4)\right]$

$= 2x - 12x^2 + 48x^3 - 160x^3 + O(x^5)$ //

(b) VALID FOR $|2x| < 1$
$|x| < \frac{1}{2}$ IE $-\frac{1}{2} < x < \frac{1}{2}$ //

Question 11 (***)

$$y = \sqrt{4-12x}, \quad -\frac{1}{3} < x < \frac{1}{3}.$$

a) Find the binomial expansion of y in ascending powers of x up and including the term in x^3, writing all coefficients in their simplest form.

b) Hence find the coefficient of x^2 in the expansion of

$$(12x-4)(4-12x)^{\frac{1}{2}}.$$

$\boxed{}$, $\boxed{y = 2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O\left(x^4\right)}$, $\boxed{-27}$

(a) $y = \sqrt{4-12x} = (4-12x)^{\frac{1}{2}} = 4^{\frac{1}{2}}(1-3x)^{\frac{1}{2}} = 2(1-3x)^{\frac{1}{2}}$

$= 2\left[1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-3x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}(-3x)^3 + O(x^4)\right]$

$= 2\left[1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + O(x^4)\right]$

$= 2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O(x^4)$

(b) $(12x-4)(4-12x)^{\frac{1}{2}}$

$(12x-4)\left(2 - 3x - \frac{9}{4}x^2 - \frac{27}{8}x^3 + O(x^4)\right)$

$-36x^2$ $\quad 9x^2$

$\therefore -27x^2$

$\therefore -27$

Question 12 (***)

The binomial $(1+x)^{-\frac{1}{2}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

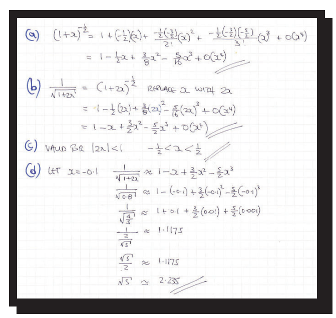
a) Find the series expansion of $(1+x)^{-\frac{1}{2}}$ up and including the term in x^3.

b) Use the expansion of part (a) to find the expansion of $\dfrac{1}{\sqrt{1+2x}}$, up and including the term in x^3.

c) State the range of values of x for which the expansion of $\dfrac{1}{\sqrt{1+2x}}$ is valid.

d) Use the expansion of $\dfrac{1}{\sqrt{1+2x}}$ with $x=-0.1$ to show that $\sqrt{5} \approx 2.235$.

$$\boxed{1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+O\left(x^4\right)}, \quad \boxed{1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+O\left(x^4\right)}, \quad \boxed{-\frac{1}{2}<x<\frac{1}{2}}$$


(a) $(1+x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(x)+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}(x)^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(x)^3+O(x^4)$

$= 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+O(x^4)$

(b) $\frac{1}{\sqrt{1+2x}} = (1+2x)^{-\frac{1}{2}}$ REPLACE x WITH $2x$

$= 1-\frac{1}{2}(2x)+\frac{3}{8}(2x)^2-\frac{5}{16}(2x)^3+O(x^4)$

$= 1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+O(x^4)$

(c) VALID FOR $|2x|<1$ $\quad -\frac{1}{2}<x<\frac{1}{2}$

(d) LET $x=-0.1$ $\quad \frac{1}{\sqrt{1+2x}} \approx 1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$

$\frac{1}{\sqrt{0.8}} \approx 1-(-0.1)+\frac{3}{2}(-0.1)^2-\frac{5}{2}(-0.1)^3$

$\frac{1}{\sqrt{\frac{4}{5}}} \approx 1+0.1+\frac{3}{2}(0.01)+\frac{5}{2}(0.001)$

$\frac{1}{\frac{2}{\sqrt{5}}} \approx 1.1175$

$\frac{\sqrt{5}}{2} \approx 1.1175$

$\sqrt{5} \simeq 2.235$

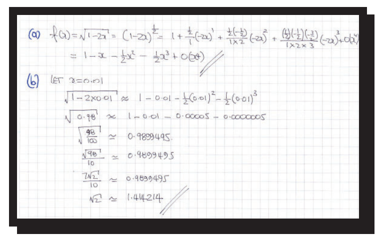
Question 13 **(***)**

$$f(x)=\sqrt{1-2x}\,, \quad |x|<\frac{1}{2}.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^3.

b) By substituting $x=0.01$ in the expansion, show that $\sqrt{2}\approx 1.414214$.

$$\boxed{f(x)=1-x-\frac{1}{2}x^2-\frac{1}{2}x^3+O\left(x^4\right)}$$



(a) $f(x)=\sqrt{1-2x}=(1-2x)^{\frac{1}{2}}=1+\frac{\frac{1}{2}}{1}(-2x)^1+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times2}(-2x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\times2\times3}(-2x)^3+O(x^4)$

$=1-x-\frac{1}{2}x^2-\frac{1}{2}x^3+O(x^4)$

(b) LET $x=0.01$

$\sqrt{1-2\times0.01}\approx 1-0.01-\frac{1}{2}(0.01)^2-\frac{1}{2}(0.01)^3$

$\sqrt{0.98}\approx 1-0.01-0.00005-0.0000005$

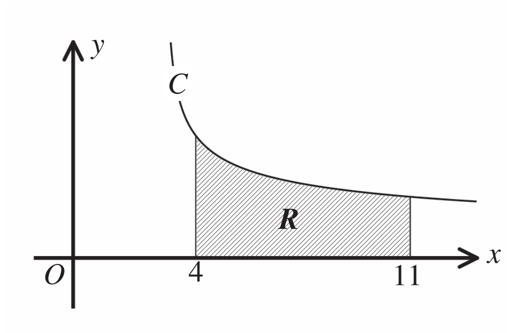
$\sqrt{\frac{98}{100}}\simeq 0.9899495$

$\frac{\sqrt{98}}{10}\simeq 0.9899495$

$\frac{7\sqrt{2}}{10}\simeq 0.9899495$

$\sqrt{2}\simeq 1.414214$

Question 4 (+)**



The figure above shows the curve C, given parametrically by

$$x = t^3 + 3 \ , \ y = \frac{2}{3t}, \ \ t > 0 .$$

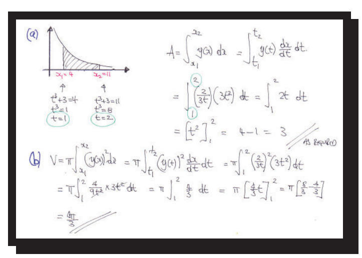
The finite region R is bounded by C, the x axis and the straight lines with equations $x = 4$ and $x = 11$.

a) Show that the area of R is 3 square units.

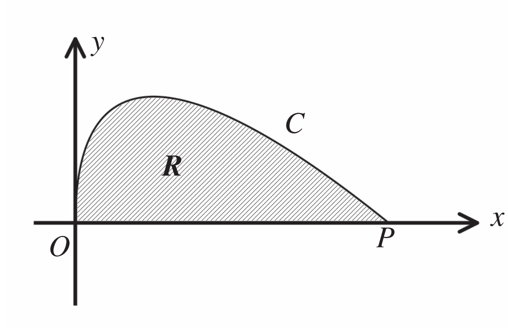
The region R is revolved in the x axis by 2π radians to form a solid of revolution S.

b) Find the volume of S.

$$\boxed{V = \frac{4\pi}{3}}$$



Question 5 (***)



The figure above shows the curve C, given parametrically by

$$x = 6t^2, \quad y = t - t^2, \quad t \geq 0.$$

The curve meets the x axis at the origin O and at the point P.

a) Show that the x coordinate of P is 6.

The finite region R, bounded by C and the x axis, is revolved in the x axis by 2π radians to form a solid of revolution, whose volume is denoted by V.

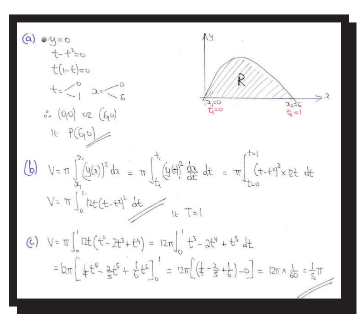
b) Show clearly that

$$V = \pi \int_0^T 12t\left(t - t^2\right)^2 \, dt,$$

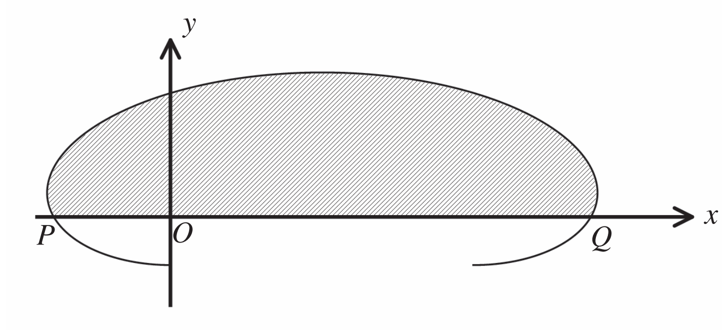
stating the value of T.

c) Hence find the value of V.

$\boxed{}$, $\boxed{T = 1}$, $\boxed{V = \frac{\pi}{5}}$



Question 11 **(***+)**



The figure above shows a curve known as a re-entrant cycloid, with parametric equations

$$x = \theta - 4\sin\theta, \quad y = 1 - 2\cos\theta \;, \quad 0 \le \theta \le 2\pi \,.$$

The curve crosses the x axis at the points P and Q.

a) Find the value of θ at the points P and Q.

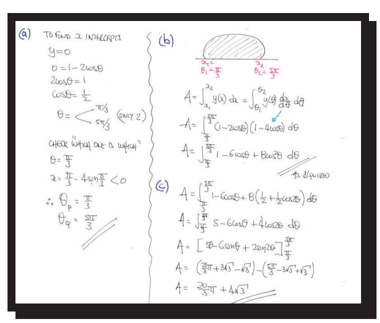
b) Show that the area of the finite region bounded by the curve and the x axis, shown shaded in the figure above, is given by the integral

$$\int_{\theta_1}^{\theta_2} 1 - 6\cos\theta + 8\cos^2\theta \; d\theta \,,$$

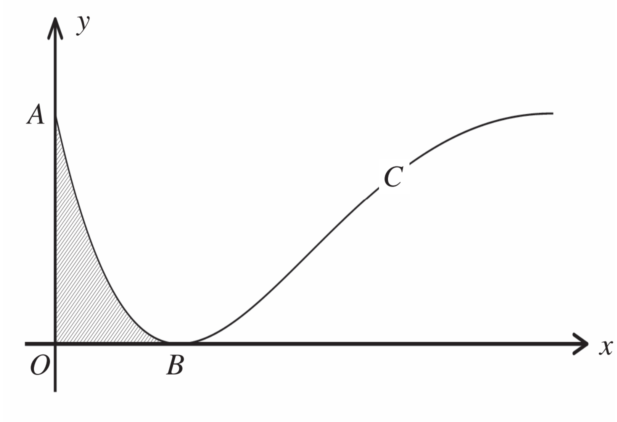
where θ_1 and θ_2 must be stated.

c) Find an exact value for the above integral.

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$, $\boxed{\theta_1 = \frac{\pi}{3}, \theta_2 = \frac{5\pi}{3}}$, $\boxed{\frac{20\pi}{3} + 4\sqrt{3}}$



Question 12 (***+)



The figure above shows the curve C, with parametric equations

$$x = t^2, \quad y = 1 + \cos t \ , \ 0 \le t \le 2\pi .$$

The curve meets the coordinate axes at the points A and B.

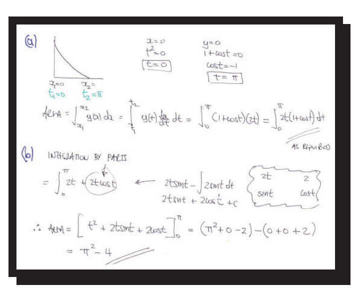
a) Show that the area of the shaded region bounded by C and the coordinate axes is given by the integral

$$\int_{t_1}^{t_2} 2t(1+\cos t) \ dt ,$$

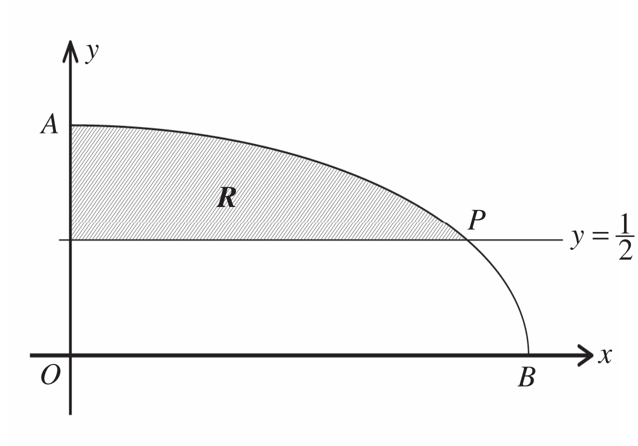
where t_1 and t_2 are constants to be stated.

b) Evaluate the above parametric integral to find an exact value for the area of the shaded region.

$\boxed{t_1 = 0, \ t_2 = \pi}$, $\boxed{\text{area} = \pi^2 - 4}$



Question 16 (****)



The figure above shows the curve C, with parametric equations

$$x = 4\cos\theta, \quad y = \sin\theta \ , \ 0 \le \theta \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B. The straight line with equation $y = \frac{1}{2}$ meets C at the point P.

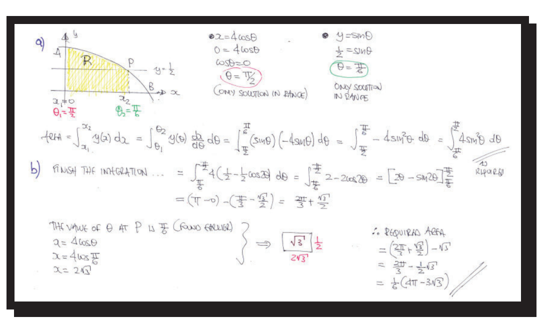
a) Show that the area under the arc of the curve between A and P, and the x axis, is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \ d\theta .$$

The shaded region R is bounded by C, the straight line with equation $y = \frac{1}{2}$ and the y axis.

b) Find an exact value for the area of R.

$$\boxed{\text{area} = \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)}$$



Question 12 (***)

A curve C is given parametrically by

$$x = 4t - 1, \quad y = \frac{5}{2t} + 10, \quad t \in \mathbb{R}, \; t \neq 0.$$

The curve C crosses the x axis at the point A.

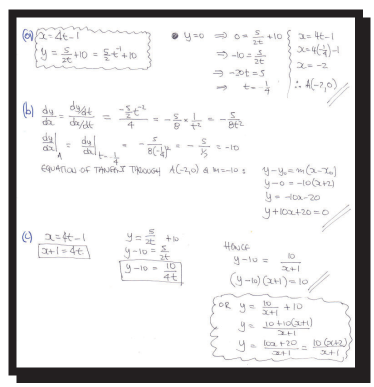
a) Find the coordinates of A.

b) Show that an equation of the tangent to C at A is

$$10x + y + 20 = 0.$$

c) Determine a Cartesian equation for C.

$$\boxed{(-2,0)}, \quad \boxed{(x+1)(y-10) = 10 \quad \text{or} \quad y = \frac{10(x+2)}{x+1}}$$



(a) $x = 4t - 1$

$y = \frac{5}{2t} + 10 = \frac{5}{2}t^{-1} + 10$

• $y = 0 \Rightarrow 0 = \frac{5}{2t} + 10$

$\Rightarrow -10 = \frac{5}{2t}$

$\Rightarrow -20t = 5$

$\Rightarrow t = -\frac{1}{4}$

$x = 4t - 1$

$x = 4(-\frac{1}{4}) - 1$

$x = -2$

$\therefore A(-2,0)$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{5}{2}t^{-2}}{4} = -\frac{5}{8} \times \frac{1}{t^2} = -\frac{5}{8t^2}$

$\left.\frac{dy}{dx}\right|_A = \left.\frac{dy}{dx}\right|_{t=-\frac{1}{4}} = -\frac{5}{8(-\frac{1}{4})^2} = -\frac{5}{\frac{1}{2}} = -10$

EQUATION OF TANGENT THROUGH $A(-2,0)$ & $m = -10$:

$y - y_0 = m(x - x_0)$

$y - 0 = -10(x+2)$

$y = -10x - 20$

$y + 10x + 20 = 0$

(c) $x = 4t - 1$

$\boxed{x + 1 = 4t}$

$y = \frac{5}{2t} + 10$

$y - 10 = \frac{5}{2t}$

$\boxed{y - 10 = \frac{10}{4t}}$

HENCE

$y - 10 = \frac{10}{x+1}$

$(y-10)(x+1) = 10$

OR $y = \frac{10}{x+1} + 10$

$y = \frac{10 + 10(x+1)}{x+1}$

$y = \frac{10x + 20}{x+1} = \frac{10(x+2)}{x+1}$

Question 13 (***)

A curve C is given parametrically by

$$x = 3t - 1, \quad y = \frac{1}{t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

Show that an equation of the normal to C at the point where C crosses the y axis is

$$y = \frac{1}{3}x + 3.$$

$\boxed{\text{proof}}$

$$\left.\begin{array}{l} x = 3t-1 \\ y = \frac{1}{t} = t^{-1} \end{array}\right\} \quad x = 0 \Rightarrow 0 = 3t - 1 \quad \boxed{t = \frac{1}{3}}$$

HENCE $y = \frac{1}{1/3} = 3 \quad \therefore (0,3)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t^{-2}}{3} = -\frac{1}{3t^2}$$

$$\left.\frac{dy}{dx}\right|_{(0,3)} = \left.\frac{dy}{dx}\right|_{t=\frac{1}{3}} = -\frac{1}{3\left(\frac{1}{3}\right)^2} = -\frac{1}{1/3} = -3 \quad \therefore \text{NORMAL GRADIENT } \frac{1}{3}$$

EQUATION OF NORMAL $\Rightarrow$

$$y - y_0 = m(x - x_0)$$
$$y - 3 = \frac{1}{3}(x - 0)$$
$$y - 3 = \frac{1}{3}x \quad \therefore y = \frac{1}{3}x + 3$$

AS REQUIRED

Question 24 (***+)

A curve C is defined by the parametric equations

$$x = \cos t, \quad y = \cos 2t \ , \ 0 \le t \le \pi .$$

a) Find $\frac{dy}{dx}$ in its simplest form.

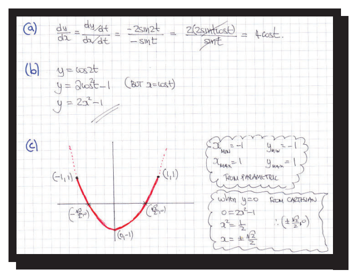
b) Find a Cartesian equation for C.

c) Sketch the graph of C.

The sketch must include

- the coordinates of the endpoints of the graph.
- the coordinates of any points where the graph meets the coordinates axes.

$\boxed{\frac{dy}{dx} = 4\cos t}$, $\boxed{y = 2x^2 - 1}$, $\boxed{(-1,1)(1,1),(0,-1),\left(-\frac{\sqrt{2}}{2},0\right)\left(\frac{\sqrt{2}}{2},0\right)}$



Question 25 (*+)**

A curve C is given by the parametric equations

$$x = \frac{3t-2}{t-1}, \quad y = \frac{t^2-2t+2}{t-1}, \quad t \in \mathbb{R}, \quad t \neq 1.$$

a) Show clearly that

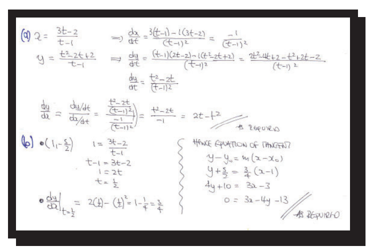
$$\frac{dy}{dx} = 2t - t^2.$$

The point $P\left(1, -\frac{5}{2}\right)$ lies on C.

b) Show that the equation of the tangent to C at the point P is

$$3x - 4y - 13 = 0.$$

$\boxed{\text{proof}}$



(a) $x = \frac{3t-2}{t-1} \implies \frac{dx}{dt} = \frac{3(t-1) - 1(3t-2)}{(t-1)^2} = \frac{-1}{(t-1)^2}$

$y = \frac{t^2-2t+2}{t-1} \implies \frac{dy}{dt} = \frac{(t-1)(2t-2) - 1(t^2-2t+2)}{(t-1)^2} = \frac{2t^2-4t+2-t^2+2t-2}{(t-1)^2}$

$\frac{dy}{dt} = \frac{t^2-2t}{(t-1)^2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t^2-2t}{(t-1)^2}}{\frac{-1}{(t-1)^2}} = \frac{t^2-2t}{-1} = 2t - t^2$ // AS REQUIRED

(b) • $\left(1, -\frac{5}{2}\right)$ $\quad 1 = \frac{3t-2}{t-1}$, $\quad t-1 = 3t-2$, $\quad 1 = 2t$, $\quad t = \frac{1}{2}$

• $\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = 2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

HENCE EQUATION OF TANGENT

$y - y_0 = m(x - x_0)$

$y + \frac{5}{2} = \frac{3}{4}(x-1)$

$4y + 10 = 3x - 3$

$0 = 3x - 4y - 13$ // AS REQUIRED

Question 49 (****)

A curve C is given by the parametric equations

$$x = \sec\theta, \quad y = \ln(1 + \cos 2\theta), \quad 0 \le \theta < \frac{\pi}{2}.$$

a) Show clearly that

$$\frac{dy}{dx} = -2\cos\theta.$$

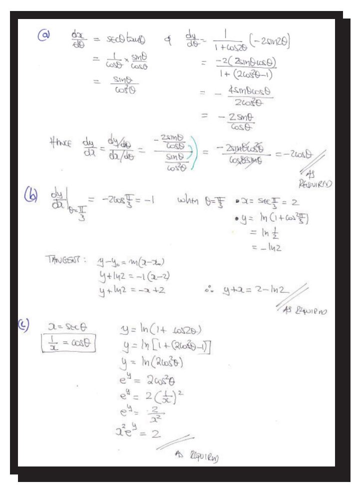
The straight line L is a tangent to C at the point where $\theta = \frac{\pi}{3}$.

b) Find an equation for L, giving the answer in the form $y + x = k$, where k is an exact constant to be found.

c) Show that a Cartesian equation of C is

$$x^2 e^y = 2.$$

$$\boxed{y + x = 2 - \ln 2}$$



(a) $\frac{dx}{d\theta} = \sec\theta\tan\theta = \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$ & $\frac{dy}{d\theta} = \frac{1}{1+\cos 2\theta}(-2\sin 2\theta) = \frac{-2(2\sin\theta\cos\theta)}{1+(2\cos^2\theta - 1)} = -\frac{4\sin\theta\cos\theta}{2\cos^2\theta} = -\frac{2\sin\theta}{\cos\theta}$

Hence $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\frac{2\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\cos^2\theta}} = \frac{-2\sin\theta\cos^2\theta}{\cos\theta\sin\theta} = -2\cos\theta$ AS REQUIRED

(b) $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -2\cos\frac{\pi}{3} = -1$ when $\theta = \frac{\pi}{3}$ • $x = \sec\frac{\pi}{3} = 2$ • $y = \ln(1+\cos\frac{2\pi}{3}) = \ln\frac{1}{2} = -\ln 2$

TANGENT: $y - y_0 = m(x - x_0)$

$y + \ln 2 = -1(x-2)$

$y + \ln 2 = -x + 2$ $\therefore y + x = 2 - \ln 2$ AS REQUIRED

(c) $x = \sec\theta$

$\boxed{\frac{1}{x} = \cos\theta}$

$y = \ln(1 + \cos 2\theta)$

$y = \ln[1 + (2\cos^2\theta - 1)]$

$y = \ln(2\cos^2\theta)$

$e^y = 2\cos^2\theta$

$e^y = 2\left(\frac{1}{x}\right)^2$

$e^y = \frac{2}{x^2}$

$x^2 e^y = 2$ AS REQUIRED

Question 3

Carry out the following integrations:

1. $$\int \tfrac{1}{2}x\mathrm{e}^{4x}\ dx = \frac{1}{8}x\mathrm{e}^{4x} - \frac{1}{32}\mathrm{e}^{4x} + C$$

2. $$\int 5x\sin 4x\ dx = -\frac{5}{4}x\cos 4x + \frac{5}{16}\sin 4x + C$$

3. $$\int (2x+1)\cos 2x\ dx = \frac{1}{2}(2x+1)\sin 2x + \frac{1}{2}\cos 2x + C$$

4. $$\int -3x\cos 4x\ dx = -\frac{3}{4}x\sin 4x - \frac{3}{16}\cos 4x + C$$

5. $$\int x^2\mathrm{e}^{-2x}\ dx = -\frac{1}{2}x^2\mathrm{e}^{-2x} - \frac{1}{2}x\mathrm{e}^{-2x} - \frac{1}{4}\mathrm{e}^{-2x} + C$$

6. $$\int x^2\sin 5x\ dx = -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C$$

7. $$\int x^2\cos\tfrac{1}{3}x\ dx = 3x^2\sin\tfrac{1}{3}x + 18x\cos\tfrac{1}{3}x - 54\sin\tfrac{1}{3}x + C$$

8. $$\int \tfrac{1}{2}x^3\ln x\ dx = \frac{1}{8}x^4\ln x - \frac{1}{32}x^4 + C$$

9. $$\int x\ln 3x\ dx = \frac{1}{2}x^2\ln 3x - \frac{1}{4}x^2 + C$$

10. $$\int \frac{\ln x}{x^3}\ dx = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

1. $$\int \tfrac{1}{2}xe^{4x}\,dx = \tfrac{1}{8}xe^{4x} - \int \tfrac{1}{8}e^{4x}\,dx$$
$$= \tfrac{1}{8}xe^{4x} - \tfrac{1}{32}e^{4x} + C$$
$$\begin{array}{ccc} \tfrac{1}{2}x & \to & \tfrac{1}{2} \\ \tfrac{1}{4}e^{4x} & \to & e^{4x} \end{array}$$

2. $$\int 5x\sin 4x\,dx = -\tfrac{5}{4}x\cos 4x - \int -\tfrac{5}{4}\cos 4x\,dx$$
$$= -\tfrac{5}{4}x\cos 4x + \int \tfrac{5}{4}\cos 4x\,dx$$
$$= -\tfrac{5}{4}x\cos 4x + \tfrac{5}{16}\sin 4x + C.$$
$$\begin{array}{cc} 5x & 5 \\ -\tfrac{1}{4}\cos 4x & \sin 4x \end{array}$$

3. $$\int (2x+1)\cos 2x\,dx = \tfrac{1}{2}(2x+1)\sin 2x - \int \sin 2x\,dx$$
$$= \tfrac{1}{2}(2x+1)\sin 2x + \tfrac{1}{2}\cos 2x + C$$
$$\begin{array}{ccc} 2x+1 & \to & 2 \\ \tfrac{1}{2}\sin 2x & \to & \cos 2x \end{array}$$

4. $$\int -3x\cos 4x\,dx = -\tfrac{3}{4}x\sin 4x - \int -\tfrac{3}{4}\sin 4x\,dx$$
$$= -\tfrac{3}{4}x\sin 4x + \int \tfrac{3}{4}\sin 4x$$
$$= -\tfrac{3}{4}x\sin 4x - \tfrac{3}{16}\cos 4x + C$$
$$\begin{array}{ccc} -3x & \to & -3 \\ \tfrac{1}{4}\sin 4x & & \cos 4x \end{array}$$

5. $$\int x^2e^{-2x}\,dx = -\tfrac{1}{2}x^2e^{-2x} - \int -xe^{-2x}\,dx$$
$$= -\tfrac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx$$
BY PARTS AGAIN
$$= -\tfrac{1}{2}x^2e^{-2x} + \left[-\tfrac{1}{2}xe^{-2x} - \int -\tfrac{1}{2}e^{-2x}\,dx\right]$$
$$= -\tfrac{1}{2}x^2e^{-2x} - \tfrac{1}{2}xe^{-2x} + \int \tfrac{1}{2}e^{-2x}\,dx$$
$$= -\tfrac{1}{2}x^2e^{-2x} - \tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} + C$$
$$\begin{array}{cc} x^2 & 2x \\ -\tfrac{1}{2}e^{-2x} & e^{-2x} \end{array} \qquad \begin{array}{cc} x & 1 \\ -\tfrac{1}{2}e^{-2x} & e^{-2x} \end{array}$$

6. $$\int x^2\sin 5x\,dx = -\tfrac{1}{5}x^2\cos 5x - \int -\tfrac{2}{5}x\cos 5x\,dx$$
$$= -\tfrac{1}{5}x^2\cos 5x + \int \tfrac{2}{5}x\cos 5x\,dx$$
BY PARTS AGAIN
$$= -\tfrac{1}{5}x^2\cos 5x + \left[\tfrac{2}{25}\sin 5x - \int \tfrac{2}{25}\sin 5x\,dx\right]$$
$$= -\tfrac{1}{5}x^2\cos 5x + \tfrac{2}{25}\sin 5x + \tfrac{2}{125}\cos 5x + C$$
$$\begin{array}{ccc} x^2 & \to & 2x \\ -\tfrac{1}{5}\cos 5x & \to & \sin 5x \end{array} \qquad \begin{array}{ccc} \tfrac{2}{5}x & \to & \tfrac{2}{5} \\ \tfrac{1}{5}\sin 5x & \to & \cos 5x \end{array}$$

7. $$\int x^2\cos \tfrac{1}{3}x\,dx = 3x^2\sin \tfrac{1}{3}x - \int 6x\sin \tfrac{1}{3}x\,dx$$
$$= 3x^2\sin \tfrac{1}{3}x - \left[-18x\cos \tfrac{1}{3}x - \int -18\cos \tfrac{1}{3}x\,dx\right]$$
$$= 3x^2\sin \tfrac{1}{3}x + 18x\cos \tfrac{1}{3}x - \int 18\cos \tfrac{1}{3}x\,dx$$
PARTS AGAIN
$$= 3x^2\sin \tfrac{1}{3}x + 18x\cos \tfrac{1}{3}x - 54\sin \tfrac{1}{3}x + C$$
$$\begin{array}{cc} x^2 & 2x \\ 3\sin \tfrac{1}{3}x & \cos \tfrac{1}{3}x \end{array} \qquad \begin{array}{ccc} 6x & \to & 6 \\ -3\cos \tfrac{1}{3}x & \to & \sin \tfrac{1}{3}x \end{array}$$

8. $$\int \tfrac{1}{2}x^3\ln x\,dx = \tfrac{1}{8}x^4\ln x - \int \tfrac{1}{8}x^4\left(\tfrac{1}{x}\right)dx$$
$$= \tfrac{1}{8}x^4\ln x - \int \tfrac{1}{8}x^3\,dx$$
$$= \tfrac{1}{8}x^4\ln x - \tfrac{1}{32}x^4 + C$$
$$\begin{array}{ccc} \ln x & \to & \tfrac{1}{x} \\ \tfrac{1}{8}x^4 & \to & \tfrac{1}{2}x^3 \end{array}$$

9. $$\int x\ln 3x\,dx = \tfrac{1}{2}x^2\ln 3x - \int \tfrac{1}{2}x^2\left(\tfrac{1}{x}\right)dx$$
$$= \tfrac{1}{2}x^2\ln 3x - \int \tfrac{1}{2}x\,dx$$
$$= \tfrac{1}{2}x^2\ln 3x - \tfrac{1}{4}x^2 + C$$
$$\begin{array}{ccc} \ln 3x & \to & \tfrac{1}{x} \\ \tfrac{1}{2}x^2 & & x \end{array}$$

10 $$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx$$
$$= -\tfrac{1}{2}x^{-2}\ln x - \int -\tfrac{1}{2}x^{-2}\left(\tfrac{1}{x}\right)dx$$
$$= -\frac{\ln x}{2x^2} + \int \tfrac{1}{2}x^{-3}\,dx$$
$$= -\frac{\ln x}{2x^2} - \tfrac{1}{4}x^{-2} + C$$
$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$
$$\begin{array}{ccc} \ln x & \to & \tfrac{1}{x} \\ -\tfrac{1}{2}x^{-2} & & x^{-3} \end{array}$$

Question 6

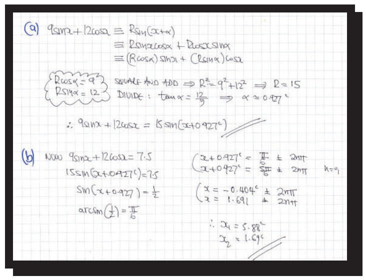
$$f(x) \equiv 9\sin x + 12\cos x\,,\ x \in \mathbb{R}\,.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

b) Hence, solve the trigonometric equation

$$9\sin x + 12\cos x = 7.5\,,\quad 0<x<2\pi\,.$$

$$\boxed{f(x) \equiv 9\sin x + 12\cos x \cong 15\sin\left(x+0.927^{\mathrm{c}}\right)}\,,\ \boxed{x \approx 1.69^{\mathrm{c}},\ 5.88^{\mathrm{c}}}$$



(a) $9\sin x + 12\cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 9$, $R\sin\alpha = 12$

SQUARE AND ADD $\Rightarrow R^2 = 9^2+12^2 \Rightarrow R = 15$

DIVIDE : $\tan\alpha = \frac{12}{9} \Rightarrow \alpha \approx 0.927^c$

$\therefore 9\sin x + 12\cos x = 15\sin(x+0.927^c)$

(b) NOW $9\sin x + 12\cos x = 7.5$

$15\sin(x+0.927^c) = 7.5$

$\sin(x+0.927) = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x + 0.927^c = \frac{\pi}{6} \pm 2n\pi$

$x + 0.927^c = \frac{5\pi}{6} \pm 2n\pi$ $\quad n = 0, 1$

$x = -0.404^c \pm 2n\pi$

$x = 1.691 \pm 2n\pi$

$\therefore x_1 = 5.88^c$

$x_2 = 1.69^c$

Question 9

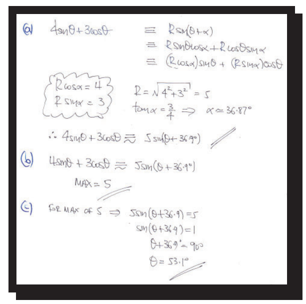
$$f(\theta) \equiv 4\sin\theta + 3\cos\theta, \ \theta \in \mathbb{R}.$$

a) Write the above expression in the form $R\sin(\theta+\alpha)$, $R>0$, $0<\alpha<90°$.

b) Write down the maximum value of $f(\theta)$.

c) Find the smallest positive value of θ for which this maximum value occurs.

$\boxed{f(\theta) \equiv 4\sin\theta + 3\cos\theta \cong 5\sin(\theta + 36.9°)}$, $\boxed{f(\theta)_{\text{max}} = 5}$, $\boxed{\theta \approx 53.1°}$



(a) $4\sin\theta + 3\cos\theta \equiv R\sin(\theta+\alpha)$
$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$
$\equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 4$
$R\sin\alpha = 3$

$R = \sqrt{4^2+3^2} = 5$
$\tan\alpha = \frac{3}{4} \Rightarrow \alpha \simeq 36.87°$

$\therefore 4\sin\theta + 3\cos\theta \approx 5\sin(\theta + 36.9°)$

(b) $4\sin\theta + 3\cos\theta \approx 5\sin(\theta + 36.9°)$
MAX = 5

(c) FOR MAX OF 5 $\Rightarrow$ $5\sin(\theta + 36.9) = 5$
$\sin(\theta + 36.9) = 1$
$\theta + 36.9° = 90°$
$\theta = 53.1°$

Question 10

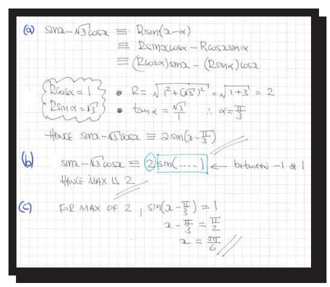
$$f(x) \equiv \sin x - \sqrt{3}\cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Write down the maximum value of $f(x)$.

c) Find the smallest positive value of x for which this maximum value occurs.

$$\boxed{f(x) \equiv \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x - \frac{\pi}{3}\right)}, \quad \boxed{f(x)_{\text{max}} = 2}, \quad \boxed{x = \frac{5\pi}{6}}$$



Question 15

$$f(x) \equiv 3\sin x + \cos x, \ x \in \mathbb{R}$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

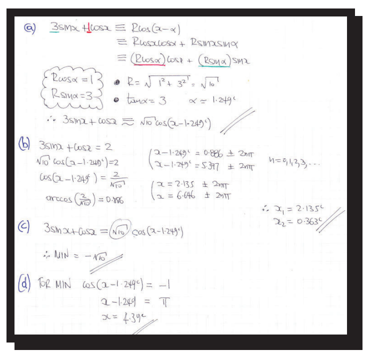
b) Solve the equation

$$f(x) = 2 \text{ for } 0 < x < 2\pi .$$

c) Write down the minimum value of $f(x)$.

d) Find the smallest positive value of x for which this minimum value occurs.

$\boxed{f(x) \cong \sqrt{10}\cos\left(x - 1.249^{c}\right)}$, $\boxed{x = 0.363^{c}, 2.135^{c}}$, $\boxed{f(x)_{\text{min}} = -\sqrt{10}}$, $\boxed{x = 4.391^{c}}$



Question 17

$$f(x) \equiv 2\sin x + 2\cos x, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) State the minimum and the maximum value of …

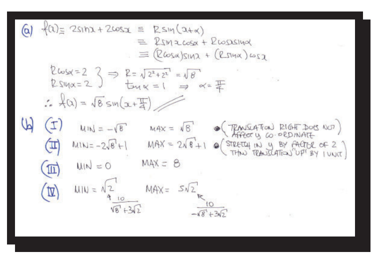
i. … $y = f\left(x - \frac{\pi}{2}\right)$.

ii. … $y = 2f(x)+1$.

iii. … $y = \left[f(x)\right]^2$.

iv. … $y = \dfrac{10}{f(x)+3\sqrt{2}}$.

$\boxed{f(x) \equiv \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)}$, $\boxed{\left[-\sqrt{8}, \sqrt{8}\right]}$, $\boxed{\left[-2\sqrt{8}+1, 2\sqrt{8}+1\right]}$, $\boxed{[0,8]}$, $\boxed{\left[\sqrt{2}, 5\sqrt{2}\right]}$


(a) $f(x) \equiv 2\sin x + 2\cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$\left.\begin{array}{l} R\cos\alpha = 2 \\ R\sin\alpha = 2 \end{array}\right\} \Rightarrow \begin{array}{l} R = \sqrt{2^2+2^2} = \sqrt{8} \\ \tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} \end{array}$

$\therefore f(x) = \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)$

(b) (I) MIN $= -\sqrt{8}$ MAX $= \sqrt{8}$ (TRANSLATION RIGHT DOES NOT AFFECT y CO-ORDINATE)

(II) MIN $= -2\sqrt{8}+1$ MAX $= 2\sqrt{8}+1$ (STRETCH IN y BY FACTOR OF 2 THEN TRANSLATION "UP" BY 1 UNIT)

(III) MIN $= 0$ MAX $= 8$

(IV) MIN $= \sqrt{2}$ MAX $= 5\sqrt{2}$

$\frac{10}{\sqrt{8}+3\sqrt{2}}$ $\frac{10}{-\sqrt{8}+3\sqrt{2}}$

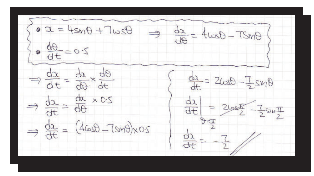
Question 7 (***)

$$x = 4\sin\theta + 7\cos\theta .$$

The value of θ is increasing at the constant rate of 0.5, in suitable units.

Find the rate at which x is changing, when $\theta = \dfrac{\pi}{2}$.

$\boxed{}$, $\boxed{-\dfrac{7}{2}}$



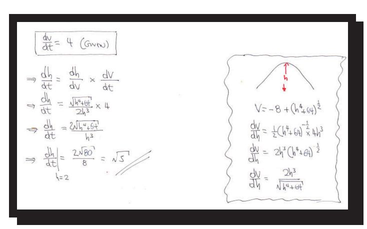
Question 8 (***)

Fine sand is dropping on a horizontal floor at the constant rate of $4 \text{ cm}^3\text{s}^{-1}$ and forms a pile whose volume, $V \text{ cm}^3$, and height, h cm, are connected by the formula

$$V = -8 + \sqrt{h^4 + 64} .$$

Find the rate at which the height of the pile is increasing, when the height of the pile has reached 2 cm.

$\boxed{}$, $\boxed{\sqrt{5} \approx 2.24 \text{ cm s}^{-1}}$



Question 9 (***)

An oil spillage on the surface of the sea remains circular at all times.

The radius of the spillage, r km, is increasing at the constant rate of 0.5 km h^{-1}.

a) Find the rate at which the area of the spillage, A km^2, is increasing, when the circle's radius has reached 10 km.

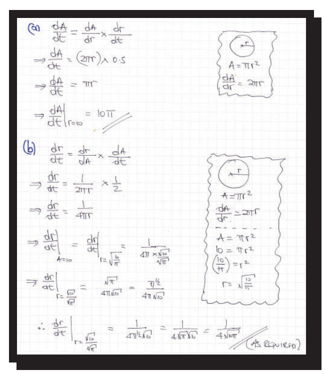
A different oil spillage on the surface of the sea also remains circular at all times.

The area of this spillage, A km^2, is increasing at the rate of $0.5 \text{ km}^2 \text{ h}^{-1}$.

b) Show that when the area of the spillage has reached 10 km^2, the rate at which the radius r of the spillage is increasing is

$$\frac{1}{4\sqrt{10\pi}} \text{ km h}^{-1}.$$

$\boxed{10\pi \approx 31.4 \text{ km}^2 \text{ h}^{-1}}$



(a) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = (2\pi r) \times 0.5$

$\Rightarrow \frac{dA}{dt} = \pi r$

$\Rightarrow \left.\frac{dA}{dt}\right|_{r=10} = 10\pi$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

(b) $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \times \frac{1}{2}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{A=10} = \left.\frac{dr}{dt}\right|_{r=\sqrt{\frac{10}{\pi}}} = \frac{1}{4\pi \times \frac{\sqrt{10}}{\sqrt{\pi}}}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{\sqrt{\pi}}{4\pi\sqrt{10}} = \frac{\pi^{\frac{1}{2}}}{4\pi\sqrt{10}}$

$\therefore \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{1}{4\pi^{\frac{1}{2}}\sqrt{10}} = \frac{1}{4\sqrt{\pi}\sqrt{10}} = \frac{1}{4\sqrt{10\pi}}$ (AS REQUIRED)

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$A = \pi r^2$

$10 = \pi r^2$

$\left(\frac{10}{\pi}\right) = r^2$

$r = \sqrt{\frac{10}{\pi}}$

Question 10 (***)

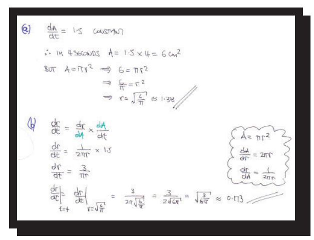
Liquid dye is poured onto a large flat cloth and forms a circular stain, the area of which grows at a steady rate of $1.5 \text{ cm}^2\text{s}^{-1}$.

Calculate, correct to three significant figures, …

a) … the radius, in cm, of the stain 4 seconds after it started forming.

b) … the rate, in cm s^{-1}, of increase of the radius of the stain after 4 seconds.

$$\boxed{\quad}, \boxed{r = \sqrt{\frac{6}{\pi}} \approx 1.38 \text{ cm}}, \boxed{\sqrt{\frac{3}{32\pi}} \approx 0.173 \text{ cm s}^{-1}}$$



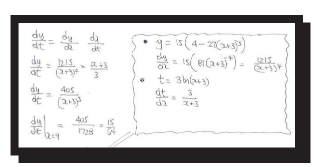
Question 11 (***)

The variables y, x and t are related by the equations

$$y = 15\left(4 - \frac{27}{(x+3)^3}\right) \quad \text{and} \quad \ln(x+3) = \frac{1}{3}t, \ x > -3.$$

Find the value of $\frac{dy}{dt}$, when $x = 9$.

$$\boxed{\quad}, \boxed{\frac{dy}{dt} = \frac{15}{64}}$$



Question 12 (***+)

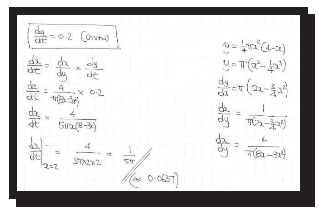
Two variables x and y are related by

$$y = \frac{1}{4}\pi x^2 (4 - x).$$

The variable y is changing with time t, at the constant rate of 0.2, in suitable units.

Find the rate at which x is changing with respect to t, when $x = 2$.

$$\boxed{\frac{1}{5\pi} \approx 0.0637}$$



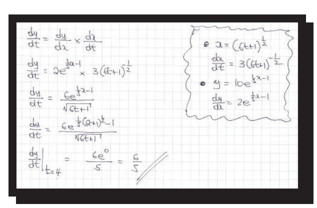
Question 13 (***+)

The variables y, x and t are related by the equations

$$y = 10e^{\frac{1}{5}x - 1} \quad \text{and} \quad x = \sqrt{6t+1}, \quad t \geq 0.$$

Find the value of $\frac{dy}{dt}$, when $t = 4$.

$$\boxed{}, \quad \boxed{\left.\frac{dy}{dt}\right|_{t=4} = \frac{6}{5}}$$



Question 14 (****)

Liquid is pouring into a container at the constant rate of $30 \text{ cm}^3\text{s}^{-1}$.

The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = 36h^2 .$$

a) Find the rate at which the height of the liquid in the container is rising when the height of the liquid reaches 3 cm.

b) Determine the rate at which the height of the liquid in the container is rising 12.5 minutes after the liquid started pouring in.

$\boxed{}$, $\boxed{\frac{5}{36} = 0.139 \text{ cm s}^{-1}}$, $\boxed{\frac{1}{60} = 0.0167 \text{ cm s}^{-1}}$