A1 Doubles Tracking Test 5 Part A

(50 marks: 60 minutes)

1. The function $f(x)$ is given by $f(x) = 5cosx + 12sinx$

   (a) Write $f(x)$ in the form $f(x) = Rcos(x - \alpha)$, where $0 \le \alpha \le 90°$ to 4s.f. **(3 marks)**

   (b) For $0 \le x \le 360°$ solve the equation $5cos2x + 12sin2x = 9$,
   giving your answers to 1 decimal place. **(4 marks)**

   (c) $g(\theta) = \frac{20}{23 - 5cos\theta - 12sin\theta}$
   Find the minimum value of $g(\theta)$, and find the value of θ closest to zero that gives this minimum value. **(3 marks)**

2. (a) Use binomial expansions to show that $\sqrt{\frac{1+3x}{1-x}} \approx 1 + 2x + 2x^3$. **(6 marks)**

   A student attempts to use this expansion to find an approximation for $\sqrt{5}$. The first few steps of the student's working are shown below.

   $$\frac{1+3x}{1-x} = 5$$

   $$1 + 3x = 5 - 5x$$

   $$x = \frac{1}{2}$$

   Substitute $x = \frac{1}{2}$ into expansion

   (b) Explain the error in the student's working. **(2 marks)**

   (c) Three other students decide to approximate $\sqrt{5}$ by using either $x = \frac{1}{5}, x = \frac{1}{6}, x = \frac{1}{17}$.

   Without approximating $\sqrt{5}$, which of these substitutions

   i) is not suitable for an approximation to $\sqrt{5}$?
   ii) gives the best approximation to $\sqrt{5}$?

   You must explain your reasoning. **(2 marks)**

3. Solve the inequality $|2x - 5| < \frac{1}{2}|x| + 1$ **(3 marks)**

PTO

4. Find the exact value of

$$\int_{\frac{1}{2}}^{2} 6x \, ln4x dx$$ **(5 marks)**

5. A spherical balloon is inflated by a pump at a rate of $12cm^3s^{-1}$. Find the rate of increase of the surface area of the balloon when the radius is 8cm.
(The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ and the surface area $S = 4\pi r^2$) **(5 marks)**

6. Find the equation of the normal to the curve $3x^2y + y^3 - 2x = 12$ at the point with coordinates (1, 2).

Give your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.
(5 marks)

7.

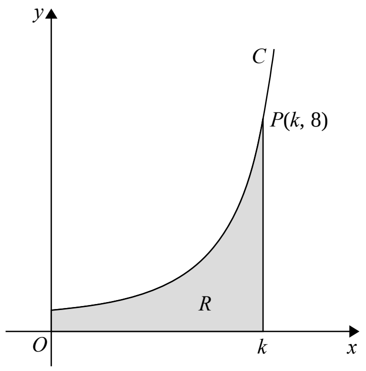


Diagram not drawn to scale

Figure 4

Figure 4 shows a sketch of part of the curve *C* with parametric equations

$$x = 3\theta\sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point *P*(*k*, 8) lies on *C*, where *k* is a constant.

(*a*) Find the exact value of *k*. **(2 marks)**

The finite region *R*, shown shaded in Figure 4, is bounded by the curve *C*, the *y*-axis, the *x*-axis and the line with equation *x* = *k*.

(*b*) Show that the area of *R* can be expressed in the form

$$\lambda \int_{\alpha}^{\beta} (\theta\sec^2\theta + \tan\theta\sec^2\theta)\,d\theta$$

where λ, α and β are constants to be determined. **(4 marks)**

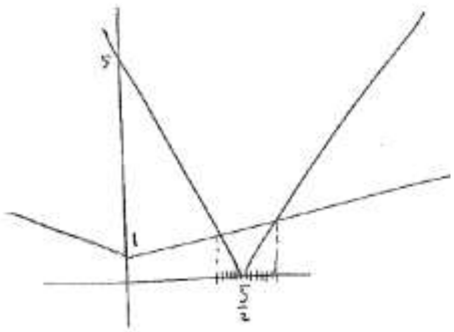
(*c*) Hence use integration to find the exact value of the area of *R*. **(6 marks)**

END OF TEST

Mark Scheme

1a	R=13 $5cosx + 12sinx \equiv 13\cos(x-\alpha)$ $= 13cos\ x\ cos\ \alpha + 13sin\ x\ sin\ \alpha$ $13sin\alpha = 12$ $\quad$ $tan\ \alpha = 2.4$ $13cos\alpha = 5$ $\quad$ $\alpha = 67.38° \ (4s.f.)$	B1 M1 (may be implied) A1
1b	$13\cos(2x - 67.38)° = 9$ $\cos(2x-67.38)° = \frac{9}{13}$ $(2x - 67.38)° = -46.19°, 46.19°, 313.81°, 406.19°$ $2x = 21.19°, 113.6°, 381.2°, 473.6°$ $x = 10.6°, 56.8°, 190.6°, 236.8°$	M1 (FT (1a)) M1 (4 sols) M1 (halving) A1
1c	$g(\theta)_{min} = \frac{20}{23+13} = \frac{5}{9}$ at $\cos(\theta - 67.38)° = -1$ $(\theta - 67.38)° = \cdots, 180°, \ldots$ $\theta = -112.6°$	B1 M1 (-1 and -180 seen) A1

2a	$(1+3x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x - \frac{9}{8}x^2 + \frac{81}{48}x^3 + \cdots$ $(1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{15}{48}x^3 + \cdots$ Multiplying together gives $1 + 2x + 2x^3$	M1 A1 M1A1 M1 A1
2b	The expansion is valid for $\lvert x \rvert < \frac{1}{3}$ and $\frac{1}{2} > \frac{1}{3}$	B1 A1
2ci 2cii	Using $x = \frac{1}{5}$ would give an approximation of $\sqrt{2}$ Using $x = \frac{1}{17}$ would give an approximation of $\frac{\sqrt{5}}{2}$ so most suitable	B1 B1

3	 $-2x + 5 = \frac{1}{2}x + 1$ $\quad$ $2x - 5 = \frac{1}{2}x + 1$ $-4x + 10 = x + 2$ $\quad$ $4x - 10 = x + 2$ $8 = 5x$ $\quad$ $3x = 12$ $x = \frac{8}{5}$ $\quad$ $x = 4$ $\boldsymbol{\frac{8}{5} < x < 4}$	M1(2 equations) A1 (2 solutions) A1

4	$\int_{\frac{1}{2}}^{2} 6x\ln 4x\, dx$ $u = \ln 4x \quad \frac{du}{dx} = \frac{1}{x}$ $v = 3x^2 \quad \frac{dv}{dx} = 6x$ $= [3x^2\ln 4x] - \int_{\frac{1}{2}}^{2} 3x^2 \times \frac{1}{x} dx$ $= \left[3x^2\ln 4x - \frac{3}{2}x^2\right]_{\frac{1}{2}}^{2}$ $= 12\ln 8 - \frac{3}{4}\ln 2 - \frac{45}{8}$	M1 M1 A1 M1 A1 A1

5	$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2 \qquad \frac{dv}{dt} = 12$ $\frac{dv}{dr} = 4\pi r^2 \qquad \frac{ds}{dr} = 8\pi r$ $\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dv} \times \frac{dv}{dt}$ $= 8\pi r \times \frac{1}{4\pi r^2} \times 12$ $= \frac{24}{r}$ When $r = 8$ $\frac{ds}{dt} = \frac{24}{8} = 3cm^3s^{-1}$	M1 (differentiate) M1 (chain rule) A1 dM1 (dep on both previous M marks) A1 (must include units)

6	$3x^2\frac{dy}{dx} + 6xy + 3y^2\frac{dy}{dx} - 2 = 0$ (1,2) $\quad 3\frac{dy}{dx} + 12 + 12\frac{dy}{dx} - 2 = 0$ $15\frac{dy}{dx} = -10$ $\frac{dy}{dx} = -\frac{2}{3}$ $y - 2 = \frac{3}{2}(x - 1)$ $2y - 4 = 3x - 3$ $3x - 2y + 1 = 0$	M1(prod &chain) A1 M1 M1 (using neg recip to find st line) A1

7	$x = 3\theta\sin\theta,\ \ y = \sec^3\theta,\quad 0 \le \theta < \frac{\pi}{2}$	
(a)	$\{\text{When } y = 8,\}\ 8 = \sec^3\theta \Rightarrow \cos^3\theta = \frac{1}{8} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ $k \text{ (or } x) = 3\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$	M1
	so $k \text{ (or } x) = \frac{\sqrt{3}\pi}{2}$	A1
		[2]
(b)	$\frac{dx}{d\theta} = 3\sin\theta + 3\theta\cos\theta$	B1
	$\left\{\int y\frac{dx}{d\theta}\{d\theta\}\right\} = \int(\sec^3\theta)(3\sin\theta + 3\theta\cos\theta)\{d\theta\}$	M1
	$= 3\int \theta\sec^2\theta + \tan\theta\sec^2\theta\ d\theta$	A1 *
	$x = 0$ and $x = k \Rightarrow \underline{\alpha = 0}$ and $\underline{\beta = \frac{\pi}{3}}$	B1
		[4]
(c)	$\left\{\int \theta\sec^2\theta\, d\theta\right\} = \theta\tan\theta - \int\tan\theta\{d\theta\}$	M1 dM1
	$= \theta\tan\theta - \ln(\sec\theta)$ **or** $= \theta\tan\theta + \ln(\cos\theta)$	A1
	$\left\{\int \tan\theta\sec^2\theta\, d\theta\right\} = \frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2u^2}$ where $u = \cos\theta$ or $\frac{1}{2}u^2$ where $u = \tan\theta$	M1 A1
	$\{\text{Area}(R)\} = \left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\tan^2\theta\right]_0^{\frac{\pi}{3}}$ **or** $\left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\sec^2\theta\right]_0^{\frac{\pi}{3}}$	
	$= \left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln 2 + \frac{3}{2}(3)\right) - (0)$ **or** $\left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln 2 + \frac{3}{2}(4)\right) - \left(\frac{3}{2}\right)$	
	$= \frac{9}{2} + \sqrt{3}\pi - 3\ln 2$ **or** $\frac{9}{2} + \sqrt{3}\pi + 3\ln\left(\frac{1}{2}\right)$ **or** $\frac{9}{2} + \sqrt{3}\pi - \ln 8$ **or** $\ln\left(\frac{1}{8}e^{\frac{9}{2} + \sqrt{3}\pi}\right)$	A1 o.e.
		[6]
		12