Example 3

Prove, from first principles, that the derivative of x^3 is $3x^2$.

$$f(x) = x^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$As h \to 0$$
, $3xh \to 0$ and $h^2 \to 0$.

From first principles' means that you have to use the definition of the derivative. You are starting your proof with a known definition, so this is an example of a proof by deduction.

$$(x+h)^3 = (x+h)(x+h)^2$$

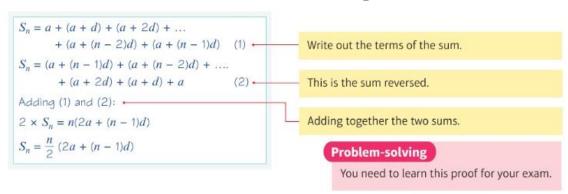
$$= (x+h)(x^2 + 2hx + h^2)$$
which expands to give $x^3 + 3x^2h + 3xh^2 + h^3$

Factorise the numerator.

Any terms containing h, h^2, h^3 , etc will have a limiting value of 0 as $h \to 0$.

Example

Prove that the sum of the first *n* terms of an arithmetic series is $\frac{n}{2}(2a + (n-1)d)$.



Example 12

A geometric series has first term a and common difference r. Prove that the sum of the first n terms of this series is given by $S_n = \frac{a(1-r^n)}{1-r}$

Let
$$S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1}$$
 Multiply by r .

$$rS_n = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n \quad (2)$$

$$(1) - (2) \text{ gives } S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

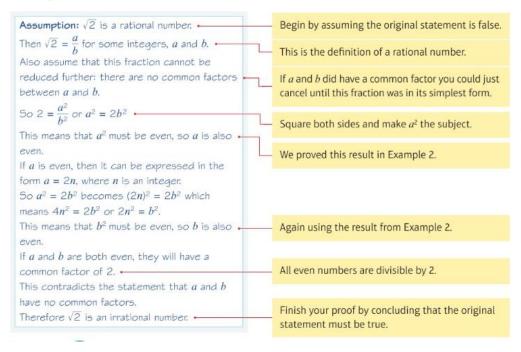
Divide by $(1-r)$.

Problem-solving

You need to learn this proof for your exam.

Example 3

Prove by contradiction that $\sqrt{2}$ is an irrational number.



Example 4

Prove by contradiction that there are infinitely many prime numbers.

