Example 3

Prove, from first principles, that the derivative of x^3 is $3x^2$.

$f(x) = x^3$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$

$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$

As $h \to 0$, $3xh \to 0$ and $h^2 \to 0$.

So $f'(x) = 3x^2$

'From first principles' means that you have to use the definition of the derivative. You are starting your proof with a known definition, so this is an example of a proof by deduction.

$(x + h)^3 = (x + h)(x + h)^2$
$= (x + h)(x^2 + 2hx + h^2)$
which expands to give $x^3 + 3x^2h + 3xh^2 + h^3$

Factorise the numerator.

Any terms containing h, h^2, h^3, etc will have a limiting value of 0 as $h \to 0$.

Example 5

Prove that the sum of the first n terms of an arithmetic series is $\frac{n}{2}(2a + (n-1)d)$.

$S_n = a + (a + d) + (a + 2d) + \dots + (a + (n-2)d) + (a + (n-1)d)$ (1)

$S_n = (a + (n-1)d) + (a + (n-2)d) + \dots + (a + 2d) + (a + d) + a$ (2)

Adding (1) and (2):

$2 \times S_n = n(2a + (n-1)d)$

$S_n = \frac{n}{2}(2a + (n-1)d)$

Write out the terms of the sum.

This is the sum reversed.

Adding together the two sums.

Problem-solving

You need to learn this proof for your exam.

Example 12

A geometric series has first term a and common difference r. Prove that the sum of the first n terms of this series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$

Let $S_n = a + ar + ar^2 + ar^3 + \dots ar^{n-2} + ar^{n-1}$ (1)

$rS_n = ar + ar^2 + ar^3 + \dots ar^{n-1} + ar^n$ (2)

(1) − (2) gives $S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(1 - r^n)}{1 - r}$

Multiply by r.

Subtract rS_n from S_n.

Take out the common factor.

Divide by $(1 - r)$.

Problem-solving

You need to learn this proof for your exam.

Example 13

Example 3

Prove by contradiction that $\sqrt{2}$ is an irrational number.

Assumption: $\sqrt{2}$ is a rational number.	Begin by assuming the original statement is false.
Then $\sqrt{2} = \frac{a}{b}$ for some integers, a and b.	This is the definition of a rational number.
Also assume that this fraction cannot be reduced further: there are no common factors between a and b.	If a and b did have a common factor you could just cancel until this fraction was in its simplest form.
So $2 = \frac{a^2}{b^2}$ or $a^2 = 2b^2$	Square both sides and make a^2 the subject.
This means that a^2 must be even, so a is also even.	We proved this result in Example 2.
If a is even, then it can be expressed in the form $a = 2n$, where n is an integer. So $a^2 = 2b^2$ becomes $(2n)^2 = 2b^2$ which means $4n^2 = 2b^2$ or $2n^2 = b^2$.	
This means that b^2 must be even, so b is also even.	Again using the result from Example 2.
If a and b are both even, they will have a common factor of 2.	All even numbers are divisible by 2.
This contradicts the statement that a and b have no common factors.	
Therefore $\sqrt{2}$ is an irrational number.	Finish your proof by concluding that the original statement must be true.

Example 4

Prove by contradiction that there are infinitely many prime numbers.

Assumption: there is a finite number of prime numbers.	Begin by assuming the original statement is false.
List all the prime numbers that exist: $p_1, p_2, p_3, \ldots, p_n$	This is a list of all possible prime numbers.
Consider the number $N = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$	This new number is one more than the product of the existing prime numbers.
When you divide N by any of the prime numbers $p_1, p_2, p_3, \ldots, p_n$ you get a remainder of 1. So none of the prime numbers $p_1, p_2, p_3, \ldots, p_n$ is a factor of N.	
So N must either be prime or have a prime factor which is not in the list of all possible prime numbers. This is a contradiction.	This contradicts the assumption that the list $p_1, p_2, p_3, \ldots, p_n$ contains all the prime numbers.
Therefore, there is an infinite number of prime numbers.	Conclude your proof by stating that the original statement must be true.