A2 Double (Further) Maths

Further Pure 1 (FP1)

Learning Pack

Name_______________

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Week beginning	Lesson		Chapter	Topic	Page
27/4		Core	9	Vectors	1,2
4/5		Core	9	Vectors	3,4,5
11/5		Core	9	Vectors	
18/5			4	Inequalities	
25/5	**Half term**				
1/6	**Progression leave**				
8/6			5	t-formula	
17/6 - 1/7					
8/7			2	Conics 1: parabolas, hyperbolas	8,9
15/7			2	Conics 1: tangents, normal, loci	10
	Summer holiday				
7/9					
14/9			2	Conics 1	11
21/9		Core	9	Consolidation: Vectors	12
28/9				Cons. t-formula inequalities	13,14
5/10					
12/10			7	Weierstrass substitution	15
19/10					
26/10	**Half term**				
2/11					
9/11	**Study leave ?**				
16/11			3	Conics 2	16
			3	Conics 2	17
23/11			3	Conics 2	18
			3	Conics 2	19
			6	Taylor series	20
30/11				Taylor series solution to DE	21
7- 14/12					
	Christmas				
4/1					
11/1			1	FP1 Vectors	22
18/1			1	FP1 Vectors	24
25/1			9	First ODE's using substitution	25
1/2			9	Second ODE's using substitution	26
8/2	**mocks**				
15/2	**Half term**				
22/2 - 1/3					
8/3			8	Simpson's rule	27
			8	Euler's method	28
			8	Midpoint method	29
			8	Euler for 2nd order DE's	30
15/3			7	L'Hôpital's rule	31
			7	Leibnitz' theorem	32

Vectors: Equation of a line	
Vector equation of a line	https://www.youtube.com/watch?v=r24zBidwago
Cartesian equation of a line	https://www.youtube.com/watch?v=puVoOw3hNGY
Vector equation of a plane	https://www.youtube.com/watch?v=T76x5B9rf5g

1) What is the vector equation of a line between two points A and B
2) What is the vector equation of a line that goes through point A and is parallel to vector $\boldsymbol{b}$

The following are points

$A = (3,1,2)$

$B = (-1,2,-1)$

$C = (2,-1,3)$

3) Find a vector equation of the line that goes through point A and B
4) Find a Cartesian equation of the line that goes through point A and B
5) Find a vector equation of a plane that contains the points A, B and C

(ans $r = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix}, \frac{x-3}{-4} = \frac{y-1}{1} = \frac{z-2}{-3}, r = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Vectors:	
Dot Product	https://www.youtube.com/watch?v=5XVFQix8tAk
Angle between two lines	https://www.youtube.com/watch?v=LyZ_xe-0HMI

What are the two formulas for $\boldsymbol{a} \cdot \boldsymbol{b}$

The following are vectors

$$\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{c} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \qquad \boldsymbol{d} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

1) Evaluate
   a. $\boldsymbol{a} \cdot \boldsymbol{b}$
   b. $\boldsymbol{c} \cdot \boldsymbol{b}$
   c. $\boldsymbol{d} \cdot \boldsymbol{a}$
   d. $\boldsymbol{b} \cdot \boldsymbol{d}$
2) Find an angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
3)

The lines l_1 and l_2 have vector equations $\mathbf{r} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + t(3\mathbf{i} - 8\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + \mathbf{k}) + s(2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ respectively.
Given that l_1 and l_2 intersect, find the size of the acute angle between the lines to one decimal place.

(Ans? -2, 6, 0, 1, 104.36 or 75.63, 68.5)

Vectors:	
Equation of a plane in scalar from	https://www.youtube.com/watch?v=aGZY6I9Rr5A
Angles between lines and planes	https://www.youtube.com/watch?v=wtpwM2y86So
Angles between two planes	https://www.youtube.com/watch?v=2LNhYZBPkDE

The following are points

$A = (3,1,2)$ $B = (-1,2,-1)$ $C = (-1,1,2)$

1) Given that C is perpendicular to the plane Π and that the plane Π contains the points A and B. Find the scalar equation of the plane Π.
2) Find the Cartesian equation of the plane Π.
3) Find the acute angle between the line *l* with equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$ and the plane with equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 2$.
4) Find the acute angle between the planes with equations $\mathbf{r}.(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) = 13$ and $\mathbf{r}.(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}) = 6$ respectively.

(ans? $r.\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 1, -x + y + 2z = 1,\ 14.9°, 78.6°$

Vectors:	
Points of intersection of lines	https://www.youtube.com/watch?v=U9NfRbvyZZM
Points of intersection of line and plane	https://www.youtube.com/watch?v=5vqzZCp1R_M

1) The lines l_1 and l_2 have vector equations
$\mathbf{r} = 3\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = -2\mathbf{j} + 3\mathbf{k} + \mu(-5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ respectively.
Show that the two lines intersect, and find the position vector of the point of intersection.

2) D
Find the coordinates of the point of intersection of the line l and the plane Π where l has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and Π has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4$.

(Ans, (5, -3, -1), (1,3,-1)

Vectors:	
Shortest distance between a point and a plane	
Shortest distance between a point and a line	https://www.youtube.com/watch?v=ZbcZTpd0VBI
Shortest distance between skew lines using the scalar product	https://www.youtube.com/watch?v=HC5YikQxwZA

The equation given in the formula book for the shortest distance between a point and a plane is given as

The perpendicular distance from the point with coordinates (α, β, γ) to the plane with equation $ax + by + cz = d$ is

$$\frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}$$

1) Find the perpendicular distance from the point with coordinates (3, 2, −1) to the plane with equation $2x - 3y + z = 5$.

2) The line l has equation $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$, and the point A has coordinates (1, 2, −1).

a Find the shortest distance between A and l.

3) The lines l_1 and l_2 have equations $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-1\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-1\end{pmatrix}$ respectively, where λ and μ are scalars.

Find the shortest distance between these two lines.

(ans $\frac{6}{\sqrt{14}}, \frac{\sqrt{29}}{3}$, $2\sqrt{2}$)

Inequalities	
	https://www.youtube.com/watch?v=r3tYTA4H61A

Question 3 (**)

Find the set of values of x that satisfy the inequality

$$\frac{5x}{x^2+4} < x.$$

$\boxed{-1 < x < 0}$, $\boxed{x > 1}$

The t formula	
Derivation	https://www.youtube.com/watch?v=jsAVMhmsFco
Using t substitution	https://www.youtube.com/watch?v=6_QqFUwKwyk

Using the substitution $t = \tan\frac{\theta}{2}$, prove that

$$\tan^2\theta + \tan\theta\sec\theta + 1 \equiv \frac{1+\sin\theta}{\cos^2\theta}$$

Conics: Parabola	
Cartesian and parametric equations of a parabola	https://youtu.be/BfFkDEnBSH0
Focus and directrix properties of a parabola	https://youtu.be/S_QIz8Xke4s

1) What is the Cartesian equation of a parabola
2) What are the parametric equations of a parabola
3) What is defined to be the focus of a parabola?
4) What is defined to be the directrix of a parabola?
5) G

   Find an equation of the parabola with:

   a focus (7, 0) and directrix $x + 7 = 0$

6) F

   Find the coordinates of the focus and an equation for the directrix of a parabola with equation:

   a $y^2 = 24x$

(ans $y^2 = 28x$, (6,0) $x + 6 = 0$

Conics: Rectangular Hyperbola	
Cartesian and Parametric equations of a rectangular hyperbola	https://youtu.be/xmEQY7RfNdg

1) What is the Cartesian equation of the rectangular hyperbola?
2) What are the parametric equations of the rectangular hyperbola?
3) What are the equations of the asymptotes of the rectangular hyperbola?

Conics:	
Tangents and normals	https://youtu.be/WD4HkEw7rJI
Loci	https://youtu.be/9Lpm5ZQZceg

1)

The point P with coordinates (75, 30) lies on the parabola C with equation $y^2 = 12x$.

Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants.

(ans, $y = \frac{x}{5} + 15$,)

Conics:	
Consolidation	

1 The ellipse E has parametric equations $x = 4\cos\theta$, $y = 9\sin\theta$.

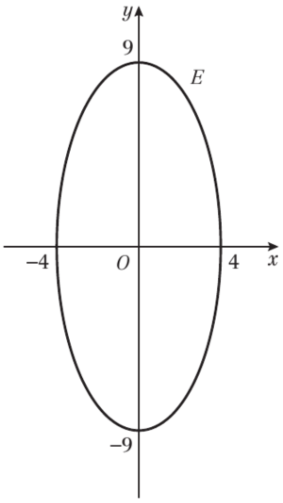
a Find a Cartesian equation of the ellipse.

b Sketch the ellipse, labelling any points of intersection with the coordinate axes.

c Find the equation of the normal to the ellipse at $P(4\cos\theta, 9\sin\theta)$.

1 a $\frac{x^2}{16} + \frac{y^2}{81} = 1$

b



c $4x\sin\theta - 9y\cos\theta = -65\cos\theta\sin\theta$

The plane Π has equation $-2x + y + z = 5$. The point P has coordinates (1, 0, 3).

a Find the shortest distance between P and Π.

The point Q is the reflection of the point P in Π.

b Find the coordinates of point Q.

a $\frac{2\sqrt{6}}{3}$ **b** $Q\left(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3}\right)$

The t formula	
Consolidation	

3 Using the substitution $t = \tan\frac{\theta}{2}$, prove that $\sin\theta + \sin\theta\cot^2\theta \equiv \operatorname{cosec}\theta$ for $\theta \neq n\pi$, $n \in \mathbb{Z}$

(4 marks)

Inequalities	
Consolidation	

7 Solve $\dfrac{4x}{|x|+2} < x$

Ans. **7** $-2 < x < 0$ or $x > 2$

		Weierstrass substitution "The sneakiest substitution in mathematics"	

Use the Weierstrass substitution to show that

$$\int \sin\theta \, d\theta = -\cos\theta$$

		Conics 2 https://en.wikipedia.org/wiki/Ellipse https://en.wikipedia.org/wiki/Hyperbola	

1) What is the standard Cartesian equation for an ellipse?
2) What are the standard parametric equations for an ellipse?
3) What is the standard Cartesian equation for a hyperbola?
4) What are the parametric equations for a hyperbola?
5) Find the equation of the tangent to the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

At the point $P(3\cos t, 2\sin t)$

(Ans 5) $3y\sin t + 2x\cos t$)

An ellipse has Cartesian equation

$$2x^2 + 3y^2 - 4x + 12y + 8 = 0.$$

Determine …

a) … the coordinates of the centre of the ellipse.

b) … the eccentricity of the ellipse.

c) … the coordinates of the foci of the ellipse.

d) … the equations of the directrices of the ellipse.

$\boxed{(1,-2)}$, $\boxed{e = \frac{\sqrt{3}}{3}}$, $\boxed{(0,-2), (2,-2)}$, $\boxed{x = -2,\ x = 4}$

Question 13 (***+)

An ellipse has equation

$$x^2 - 8x + 4y^2 + 12 = 0.$$

a) Determine the coordinates of the foci and the equations of the directrices of the ellipse.

A straight line with positive gradient passes through the origin O and **touches** the ellipse at the point A.

b) Find the coordinates of A.

$\boxed{\left(4-\sqrt{3},0\right),\ \left(4+\sqrt{3},0\right)}$, $\boxed{x = 4-\frac{4}{3}\sqrt{3},\ x = 4+\frac{4}{3}\sqrt{3}}$

Question 7 (***)

An ellipse E has Cartesian equation

$$\frac{x^2}{289}+\frac{y^2}{64}=1.$$

a) Find the coordinates of the foci of E, and the equations of its directrices.

b) Sketch the ellipse.

The point P lies on E so that PS is vertical, where S is the focus of the ellipse with positive x coordinate.

c) Show that the tangent to the ellipse at the point P meets one the directrices of the ellipse on the x axis.

$\boxed{(\pm 15,0)}$, $\boxed{x=\pm\frac{289}{15}}$

		Taylor Series Derivation - https://www.youtube.com/watch?v=kZf6phY418U Example - https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-11/v/fourth-degree-coefficient-for-taylor-polynomial	

$$y = \frac{1}{\sqrt{x}}, \quad x > 0$$

a) Find the first four terms in the Taylor expansion of y about $x = 1$.

b) Use the first three terms of the expansion with $x = \frac{8}{9}$ to show $\sqrt{2} \approx \frac{229}{162}$.

$$\boxed{y = 1 - \tfrac{1}{2}(x-1) + \tfrac{3}{8}(x-1)^2 - \tfrac{5}{16}(x-1)^3 + O\left((x-1)^4\right)}$$

		Taylor series solution to ODE’s https://www.youtube.com/watch?v=Ky5fWB0OHa4	

A curve has an equation $y = f(x)$ that satisfies the differential equation

$$\frac{dy}{dx} = x^2 - y^2 ,$$

subject to the condition $x = 0$, $y = 2$.

Find the first four terms in the expansion of $y = f(x)$ in powers of x.

$$\boxed{y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O\left(x^4\right)}$$

FP1 Vectors	
Vector product -	https://www.youtube.com/watch?v=2wTUqZa66ng
Mod of cross product -	https://www.youtube.com/watch?v=3tEdru2rwuI
Area of a triangle-	https://www.youtube.com/watch?v=lHRY5DgGdBI
Area of a parallelogram -	https://www.youtube.com/watch?v=CD0MXPhRkvQ
Volume of a tetrahedron –	https://www.youtube.com/watch?v=RVDmjbuFEUo
Volume of a parallelepiped -	https://www.youtube.com/watch?v=qpsFTTAu15M

1) Given $a = 2i - 3j$ and $b = 4i + j - k$ find $a \times b$
Verify that $a \times b$ is perpendicular to both a and b
2) Find the sine of the acute angle between the vectors $a = 2i + j + 2k$ and $b = -3j + 4k$
3) Find the area of triangle OAB, where o is the origin. A is the point with positon vector $i - j$ and B is the point with position vector $3i + 4j - 6k$
4) Find the area of the parallelogram $ABCD$, where the position vectors of A, B and D are $2i + j - k,\ 6i + 4j - 3k$ and $14i + 7j - 6k$ respectively.
5) Find the volume of a tetrahedron which has vertices at $(1,1,-1), (2,4,-1), (3,0,-2)$ and $(0,4,5)$
6) Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates $(0,0,0), (3,0,1), (1,2,0)$ and $(1,1,3)$

(Ans 1) 31+2j+14k 2) $\frac{2\sqrt{2}}{3}$ 3) $\frac{\sqrt{121}}{2}$ 4) 13 5) 6 6)17)

		FP1 Vectors Vector product equation of a straight line - https://www.youtube.com/watch?v=krc4mH9z4gE	

1) Find the vector equation of the line through the points $(1,2,-1)$ and $(3,-2,2)$ in the form $(r-1)\times b=0$
2)
   a. Find, in the form $r.n=p$, an equation of the plane which contains the line l and the point with position vector a where l has equation $r=3i+5j-2k+\lambda(-i+2j-k)$ and $a=4i+3j+k$
   b. Give the equation of the plane in Cartesian form

(Ans 2) $r.(4i+2j)=22$)

		ODE Substitution https://www.youtube.com/watch?v=iSHFmV1xxhk	

a) Use the substitution $u = y - x$ to transform the differential equation $\frac{dy}{dx} = \frac{y-x+2}{y-x+3}$ into a differential equation in u and x

b) By first solving this new equation, show that the general solution to the original equation, show that the general solution to the original equation may be written in the form

$$(y-x)^2 + 6y - 4x - 2c = 0$$

Where c is an arbitrary constant.

(Ans)

		ODE Substitution	

The differential equation

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x, \quad x \neq 0,$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}$, $\frac{dy}{dx} = \frac{1}{2}$ at $x = 1$.

a) Show that the substitution $v = \frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx} + \frac{2v}{x} = 3.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{1}{2}\left(x^2 + \frac{1}{x} + 1\right)}$$

		Simpson's rule https://www.youtube.com/watch?v=ns3k-Lz7qWU	

Simpson's rule is used to find the approximate area under a graph. Since integration between two limits also gives the area under a graph then Simpson's rule can often be used as a way of finding an approximate value of a definite integral. It is an improvement on the trapezium rule as it uses a parabola rather than a straight line between intervals as an approximation to the curve.

(i)

Use Simpson's rule with 4 intervals to estimate

$$\int_2^4 \frac{\ln x}{x}\,dx$$

(ii) Find the exact value of this integral

(iii) Hence the relative error in %: $\frac{\ldots\ldots\ldots\ldots - 0.7207}{0.7207} \times 100\%$

		Euler's method for numerically solving 1st order ODE's	

- **Euler's method for finding approximate solutions to first-order differential equations uses the approximation**

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

It is often more useful to write this as an iterative formula:

$$y_{r+1} \approx y_r + h\left(\frac{dy}{dx}\right)_r, \; r = 0, 1, 2, \ldots$$

Example 1

$y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{x^2 + y}{y^2 - x}$ and the initial condition, $f(3) = -1$.

Use two iterations of Euler's method to estimate the value of $f(4)$, giving your answer correct to 2 decimal places.

$h = 0.5$ — You need to use two iterations to get from $x_0 = 3$ to $x_2 = 4$, so your step length will be 0.5.

$(x_0, y_0) = (3, -1)$

$\left(\frac{dy}{dx}\right)_0 = \frac{3^2 - 1}{(-1)^2 - 3} = -4$ — Substitute the values of x_0 and y_0 into the differential equation to find the value of $\left(\frac{dy}{dx}\right)_0$.

$y_1 \approx y_0 + h\left(\frac{dy}{dx}\right)_0$

$= -1 + 0.5 \times (-4)$

$= -3$ — Your values of x_1 and y_1 determine the starting point for the next iteration. Use the differential equation to find the gradient at (x_1, y_1).

$(x_1, y_1) = (3.5, -3)$

$\left(\frac{dy}{dx}\right)_1 = \frac{3.5^2 - 3}{(-3)^2 - 3.5} = 1.6818\ldots$

$y_2 \approx y_1 + h\left(\frac{dy}{dx}\right)_1$

$= -3 + 0.5 \times 1.6818\ldots$

$= -2.15909\ldots$

So $f(4) \approx -2.16$ (2 d.p.) — Do not round any values until your final answer.

1 Use Euler's method to estimate the value at $x = 2$ of the particular solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

which passes through the point (1, 2). Use a step length of 0.25.

Hide

1 87.3 (3 s.f.)

		Midpoint method	

- **The midpoint method for finding approximate solutions to first-order differential equations uses the formula**

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$$

It is often more useful to write this as an iterative formula:

$$y_{r+1} \approx y_{r-1} + 2h\left(\frac{dy}{dx}\right)_r, \ r = 0, 1, 2, \ldots$$

Example 2

Use the midpoint formula with a step length of 0.25 to estimate the value at $x = 0.5$ of the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{xy + y}{y^2 + x^2}$$

which passes through the point (0, 2). Give your answer correct to 4 decimal places.

$y_2 \approx y_0 + 2h\left(\frac{dy}{dx}\right)_1$ — Your initial condition will be (x_0, y_0), so rewrite the midpoint formula using y_2 and y_0.

$x_0 = 0,\ y_0 = 2,\ h = 0.25$

$x_1 = 0.25$

$x_2 = 0.5$

Watch out Write down the information you know. You can't calculate $\left(\frac{dy}{dx}\right)_1$ without a value for y_1, so the first step in your method is to use Euler's method to find y_1.

$\left(\frac{dy}{dx}\right)_0 = \frac{0 \times 2 + 2}{2^2 + 0^2} = \frac{1}{2}$

$y_1 \approx y_0 + \left(\frac{dy}{dx}\right)_0 h$

$= 2 + \frac{1}{2} \times 0.25$

$= 2.125$

$\left(\frac{dy}{dx}\right)_1 = \frac{0.25 \times 2.125 + 2.125}{2.125^2 + 0.25^2} = 0.58020\ldots$ — Calculate $\left(\frac{dy}{dx}\right)_1$ using your value of y_1.

$y_2 \approx y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$= 2 + 2 \times 0.25 \times 0.58020\ldots$

$= 2.2901$ (4 d.p.) — Use the midpoint formula to calculate y_2.

1 A particular solution to the differential equation $\frac{dy}{dx} = x^3 - y^2$ passes through the point (2, 2).

a Taking $(x_0, y_0) = (2, 2)$ and $x_1 = 2.25$, apply Euler's method once to obtain a value for y_1.

b Apply the midpoint method once to obtain an approximate value for the solution to the differential equation at $x = 2.5$.

1 a 3

		Euler's method for numerically solving 2nd order ODE's	

- **Euler's method can be extended to find approximate solutions to second-order differential equations using the formula**

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

It is often more useful to write this as an iterative formula:

$$y_{r+1} \approx 2y_r - y_{r-1} + h^2\left(\frac{d^2y}{dx^2}\right)_r, \ r = 0, 1, 2, \ldots$$

If a second-order differential equation is of the form $\frac{d^2y}{dx^2} = f(x, y)$, you can use a single application of Euler's method to find y_1 before applying the above iterative formula.

$\frac{d^2x}{dt^2} - \sin(x + t) = 0$. When $t = 0$, $x = -1$ and $\frac{dx}{dt} = 3$.

Use the approximations $\left(\frac{dx}{dt}\right)_0 \approx \frac{x_1 - x_0}{h}$ and $\left(\frac{d^2x}{dt^2}\right)_0 \approx \frac{x_1 - 2x_0 + x_{-1}}{h^2}$ to obtain estimates for x at $t = 0.1$ and $t = 0.2$, giving your answers correct to 4 decimal places.

$x_0 = -1$, $\left(\frac{dx}{dt}\right)_0 = 3$, $h = 0.1$

$$x_1 \approx x_0 + h\left(\frac{dx}{dt}\right)_0 = -1 + 0.1 \times 3 = -0.7$$

You need two values of x to substitute into the approximation for $\frac{d^2x}{dt^2}$. You are given x_0 and you can use Euler's formula to find x_1.

$$\left(\frac{d^2x}{dt^2}\right)_1 = \sin(x_1 + t_1) = \sin(-0.7 + 0.1) = -0.5646\ldots$$

Rearrange the original equation to evaluate $\left(\frac{d^2x}{dt^2}\right)_1$, using the value of x_1 you have just found.

$$x_2 \approx 2x_1 - x_0 + h^2\left(\frac{d^2x}{dt^2}\right)_1 = 2(-0.7) - (-1) + 0.1^2(-0.5646\ldots) = -0.4056 \text{ (4 d.p.)}$$

Watch out Be careful with the index numbers when using the approximation formula for $\frac{d^2x}{dt^2}$. The index number of $\frac{d^2x}{dt^2}$ should be **one less** than the index number of the value you are approximating.

If a second-order differential equation includes a term in $\frac{dy}{dx}$, you will also need to make use of the approximation $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$

1 Use the approximations $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ and $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$ to obtain estimates for y_1, y_2 and y_3 for the following differential equations. In each case the initial conditions and step length are given.

a $\frac{d^2y}{dx^2} = x + y - 1$, given that when $x = 2$, $y = 4$ and $\frac{dy}{dx} = 1$, $h = 0.1$

1 a 4.1, 4.252, 4.45852

		L'Hôpital's rule https://www.youtube.com/watch?v=Sp0G-VggAoU	

1 Use L'Hospital's rule to calculate the following limits.

a $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 + 3x - 4}$ **b** $\lim_{x \to 4} \dfrac{x - 4}{\sqrt{x} - 2}$

1 **a** $\frac{2}{5}$ **b** 4

Guillaume de l'Hôpital

		Leibnitz' theorem https://youtu.be/tvszK9x2-1Q	

Note: pronounce Leibni(t)z as 'Lipenits' not as in the video!

Use Leibnitz's theorem to find *i*) $\frac{d^2}{dx^2}\left(\frac{e^x}{\cos x}\right)$ *ii*) $\frac{d^3}{dx^3}(x^3 \sin x)$