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# A Level Mathematics

## Revision

# Contents

# 1. Proof

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **1**<br><br>**Proof** | 1.1 | **Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including:** | **Examples of proofs:** |
| | | **Proof by deduction** | **Proof by deduction**<br><br>**e.g. using completion of the square, prove that $n^2 - 6n + 10$ is positive for all values of $n$ or, for example, differentiation from first principles for small positive integer powers of $x$** or proving results for arithmetic and geometric series. **This is the most commonly used method of proof throughout this specification** |
| | | **Proof by exhaustion** | **Proof by exhaustion**<br><br>**This involves trying all the options. Suppose $x$ and $y$ are odd integers less than 7. Prove that their sum is divisible by 2.** |
| | | **Disproof by counter example** | **Disproof by counter example**<br><br>**e.g. show that the statement "$n^2 - n + 1$ is a prime number for all values of $n$" is untrue** |
| | | Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs). | |

# 1. Proof

1 Prove that $\dfrac{x-y}{\sqrt{x}-\sqrt{y}} \equiv \sqrt{x}+\sqrt{y}$.

P 2 Prove that the sum of two consecutive positive odd numbers less than ten gives an even number.

P 3 Prove that the statement '$n^2 - n + 3$ is a prime number for all values of $n$' is untrue.

E/P 4 Prove by contradiction that $\sqrt{\frac{1}{2}}$ is an irrational number. **(5 marks)**

P 5 Prove that if $q^2$ is an irrational number then $q$ is an irrational number.

P 6 Use proof by contradiction to show that there exist no integers $a$ and $b$ for which $21a + 14b = 1$.

**Hint** Assume the opposite is true, and then divide both sides by the highest common factor of 21 and 14.

# 1. Proof

## Answers

1 $\dfrac{x-y}{(\sqrt{x}-\sqrt{y})} \times \dfrac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \dfrac{(x-y)(\sqrt{x}+\sqrt{y})}{x-y} = \sqrt{x}+\sqrt{y}$

2 $1 + 3 =$ even, $3 + 5 =$ even, $5 + 7 =$ even, $7 + 9 =$ even

3 For example when $n = 6$

4 Assume $\sqrt{\frac{1}{2}}$ is a rational number.

Then $\sqrt{\frac{1}{2}} = \dfrac{a}{b}$ for some integers $a$ and $b$.

Further assume that this fraction is in its simplest terms: there are no common factors between $a$ and $b$.

So $0.5 = \dfrac{a^2}{b^2}$ or $2a^2 = b^2$.

Therefore $b^2$ must be a multiple of 2.
We know that this means $b$ must also be a multiple of 2.
Write $b = 2c$, which means $b^2 = (2c)^2 = 4c^2$.
Now $4c^2 = 2a^2$, or $2c^2 = a^2$.
Therefore $a^2$ must be a multiple of 2, which implies $a$ is also a multiple of 2.
If $a$ and $b$ are both multiples of 2, this contradicts the statement that there are no common factors between $a$ and $b$.
Therefore, $\sqrt{\frac{1}{2}}$ is an irrational number.

5 Assume there exists a rational number $q$ such that $q^2$ is irrational.

So write $q = \dfrac{a}{b}$ where $a$ and $b$ are integers.

$q^2 = \dfrac{a^2}{b^2}$

As $a$ and $b$ are integers $a^2$ and $b^2$ are integers.
So $q^2$ is rational.

6 Assumption: there exists integers $a$ and $b$ such that $21a + 14b = 1$.
Since 21 and 14 are multiples of 7, divide both sides by 7.
So now $3a + 2b = \frac{1}{7}$
$3a$ is also an integer. $2b$ is also an integer.
The sum of two integers will always be an integer, so $3a + 2b =$ 'an integer'.
This contradicts the statement that $3a + 2b = \frac{1}{7}$ .
Therefore there exists no integers a and b for which $21a + 14b = 1$.

# 2. Algebra and Functions

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **2**<br>**Algebra and functions** | 2.1 | Understand and use the laws of indices for all rational exponents. | $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$<br>The equivalence of $a^{\frac{m}{n}}$ and $\sqrt[n]{a^m}$ should be known. |
| | 2.2 | Use and manipulate surds, including rationalising the denominator. | Students should be able to simplify algebraic surds using the results<br>$\left(\sqrt{x}\right)^2 = x$, $\sqrt{xy} = \sqrt{x}\sqrt{y}$ and<br>$\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) = x - y$ |
| | 2.3 | Work with quadratic functions and their graphs. | The notation $\mathrm{f}(x)$ may be used |
| | | The discriminant of a quadratic function, including the conditions for real and repeated roots. | Need to know and to use<br>$b^2 - 4ac > 0$, $b^2 - 4ac = 0$ and<br>$b^2 - 4ac < 0$ |
| | | Completing the square. | $ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2 + \left(c - \dfrac{b^2}{4a}\right)$ |
| | | Solution of quadratic equations | Solution of quadratic equations by factorisation, use of the formula, use of a calculator and completing the square. |
| | | including solving quadratic equations in a function of the unknown. | These functions could include powers of $x$, trigonometric functions of $x$, exponential and logarithmic functions of $x$. |
| | 2.4 | Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation. | The quadratic may involve powers of 2 in one unknown or in both unknowns,<br>e.g. solve $y = 2x + 3$, $y = x^2 - 4x + 8$<br>or<br>$2x - 3y = 6$, $x^2 - y^2 + 3x = 50$ |

# 2. Algebra and Functions

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **2**<br><br>**Algebra and functions**<br><br>*continued* | 2.5 | **Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically,** | **e.g. solving**<br>$ax + b > cx + d$,<br>$px^2 + qx + r \geq 0$,<br>$px^2 + qx + r < ax + b$<br>**and interpreting the third inequality as the range of $x$ for which the curve $y = px^2 + qx + r$ is below the line with equation $y = ax + b$** |
| | | **including inequalities with brackets and fractions.** | **These would be reducible to linear or quadratic inequalities**<br><br>**e.g.** $\frac{a}{x} < b$ **becomes** $ax < bx^2$ |
| | | **Express solutions through correct use of 'and' and 'or', or through set notation.** | **So, e.g.** $x < a$ **or** $x > b$ **is equivalent to** $\{x : x < a\} \cup \{x : x > b\}$<br>**and** $\{x : c < x\} \cap \{x : x < d\}$ **is equivalent to** $x > c$ **and** $x < d$ |
| | | **Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.** | **Shading and use of dotted and solid line convention is required.** |
| | 2.6 | **Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.** | **Only division by $(ax + b)$ or $(ax - b)$ will be required. Students should know that if $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.**<br><br>**Students may be required to factorise cubic expressions such as $x^3 + 3x^2 - 4$ and $6x^3 + 11x^2 - x - 6$.** |
| | | Simplify rational expressions, including by factorising and cancelling, and algebraic division (by linear expressions only). | Denominators of rational expressions will be linear or quadratic,<br><br>e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$ |

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **2**<br><br>**Algebra and functions**<br><br>*continued* | 2.7 | **Understand and use graphs of functions; sketch curves defined by simple equations including polynomials** | **Graph to include simple cubic and quartic functions,**<br><br>**e.g. sketch the graph with equation $y = x^2(2x - 1)^2$** |
| | | The modulus of a linear function. | Students should be able to sketch the graphs of $y = \lvert ax + b \rvert$<br><br>They should be able to use their graph.<br><br>For example, sketch the graph with equation $y = \lvert 2x - 1 \rvert$ and use the graph to solve the equation $\lvert 2x - 1 \rvert = x$ or the inequality $\lvert 2x - 1 \rvert > x$ |
| | | $y = \frac{a}{x}$ **and** $y = \frac{a}{x^2}$<br><br>**(including their vertical and horizontal asymptotes)** | **The asymptotes will be parallel to the axes e.g. the asymptotes of the curve with equation** $y = \frac{2}{x+a} + b$ **are the lines with equations** $y = b$ **and** $x = -a$ |
| | | **Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.** | |
| | | **Understand and use proportional relationships and their graphs.** | **Direct proportion between two variables.**<br><br>**Express relationship between two variables using proportion "$\propto$" symbol or using equation involving constant**<br><br>**e.g. the circumference of a semicircle is directly proportional to its diameter so $C \propto d$ or $C = kd$ and the graph of $C$ against $d$ is a straight line through the origin with gradient $k$.** |

# 2. Algebra and Functions

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **2**<br>**Algebra and functions**<br>*continued* | 2.8 | Understand and use composite functions; inverse functions and their graphs. | The concept of a function as a one-one or many-one mapping from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$. The notation $\mathrm{f}: x \mapsto$ and $\mathrm{f}(x)$ will be used. Domain and range of functions.<br>Students should know that fg will mean 'do g first, then f' and that if $\mathrm{f}^{-1}$ exists, then<br>$\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{f}\mathrm{f}^{-1}(x) = x$<br>They should also know that the graph of<br>$y = \mathrm{f}^{-1}(x)$ is the image of the graph of<br>$y = \mathrm{f}(x)$ after reflection in the line $y = x$ |
| | 2.9 | **Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$, including sketching associated graphs:**<br>$y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$,<br>$y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$ **and combinations of these transformations** | Students should be able to find the graphs of $y = \lvert \mathrm{f}(x) \rvert$ and $y = \lvert \mathrm{f}(-x) \rvert$, given the graph of $y = \mathrm{f}(x)$.<br>**Students should be able to apply a combination of these transformations to any of the functions in the A Level specification (quadratics, cubics, quartics, reciprocal, $\frac{a}{x^2}$, $\lvert x \rvert$, $\sin x$, $\cos x$, $\tan x$, $\mathrm{e}^x$ and $a^x$) and sketch the resulting graph.**<br>Given the graph of $y = \mathrm{f}(x)$, students should be able to sketch the graph of, e.g. $y = 2\mathrm{f}(3x)$, or $y = \mathrm{f}(-x) + 1$,<br>and should be able to sketch (for example)<br>$y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$ |
| | 2.10 | Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear). | Partial fractions to include denominators such as<br>$(ax + b)(cx + d)(ex + f)$ and<br>$(ax + b)(cx + d)^2$.<br>Applications to integration, differentiation and series expansions. |
| **2**<br>**Algebra and functions**<br>*continued* | 2.11 | Use of functions in modelling, including consideration of limitations and refinements of the models. | For example, use of trigonometric functions for modelling tides, hours of sunlight, etc. Use of exponential functions for growth and decay (see Paper 1, Section 6.7). Use of reciprocal function for inverse proportion (e.g. pressure and volume). |

# 2. Algebra and Functions

E/P 1 Lynn is selling cushions as part of an enterprise project. On her first attempt, she sold 80 cushions at the cost of £15 each. She hopes to sell more cushions next time. Her adviser suggests that she can expect to sell 10 more cushions for every £1 that she lowers the price.

**a** The number of cushions sold $c$ can be modelled by the equation $c = 230 - Hp$, where £$p$ is the price of each cushion and $H$ is a constant. Determine the value of $H$. **(1 mark)**

To model her total revenue, £$r$, Lynn multiplies the number of cushions sold by the price of each cushion. She writes this as $r = p(230 - Hp)$.

**b** Rearrange $r$ into the form $A - B(p - C)^2$, where $A$, $B$ and $C$ are constants to be found. **(3 marks)**

**c** Using your answer to part **b** or otherwise, show that Lynn can increase her revenue by £122.50 through lowering her prices, and state the optimum selling price of a cushion. **(2 marks)**

E 2 Find the set of values of $x$ for which the curve with equation $y = 2x^2 + 3x - 15$ is below the line with equation $y = 8 + 2x$. **(5 marks)**

E/P 3 **a** Factorise completely $x^3 - 6x^2 + 9x$. **(2 marks)**

**b** Sketch the curve of $y = x^3 - 6x^2 + 9x$ showing clearly the coordinates of the points where the curve touches or crosses the axes. **(4 marks)**

**c** The point with coordinates $(-4, 0)$ lies on the curve with equation $y = (x - k)^3 - 6(x - k)^2 + 9(x - k)$ where $k$ is a constant.
Find the two possible values of $k$. **(3 marks)**

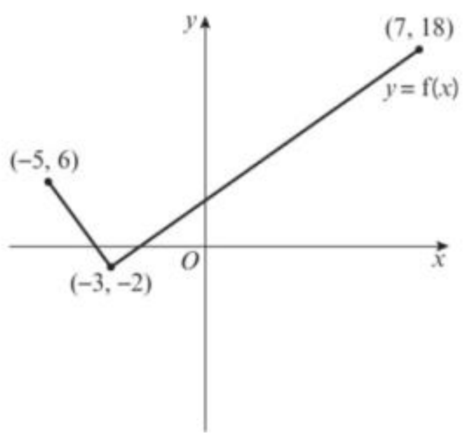
E/P 4 $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

**a** find the values of $r$ and $s$ **(6 marks)**

**b** factorise $h(x)$. **(3 marks)**

E 5 Given that $\frac{x^2 + 1}{x(x - 2)} \equiv P + \frac{Q}{x} + \frac{R}{x - 2}$, find the values of the constants $P$, $Q$ and $R$. **(5 marks)**

E/P 6 The function f has domain $-5 \leqslant x \leqslant 7$ and is linear from $(-5, 6)$ to $(-3, -2)$ and from $(-3, -2)$ to $(7, 18)$.
The diagram shows a sketch of the function.




**a** Write down the range of f. **(1 mark)**

**b** Find ff(−3). **(2 marks)**

**c** Sketch the graph of $y = |f(x)|$, marking the points at which the graph meets or cuts the axes. **(3 marks)**

The function g is defined by g: $x \mapsto x^2 - 7x + 10$.

**d** Solve the equation $fg(x) = 2$. **(3 marks)**

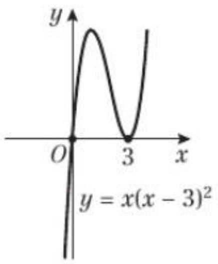
# 2. Algebra and Functions

## Answers

1 **a** $H = 10$
**b** $r = 1322.5 - 10(p - 11.5)^2$
$A = 1322.5, B = 10, C = 11.5$
**c** Old revenue is $80 \times £15 = £1200$; new revenue is £1322.50; difference is £122.50. The best selling price of a cushion is £11.50.

2 $\frac{1}{4}(-1 - \sqrt{185}) < x < \frac{1}{4}(-1 + \sqrt{185})$

3 **a** $x(x - 3)^2$
**b**

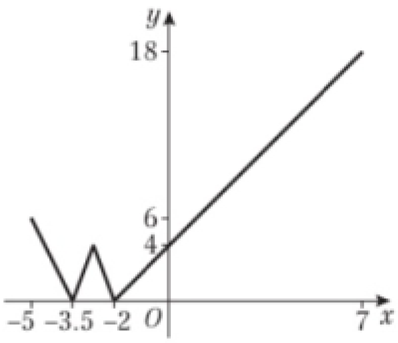



**c** −4 and −7

4 **a** $r = 3, s = 0$ **b** $x(x + 1)(x + 3)$

5 $P = 1, Q = -\frac{1}{2}, R = \frac{5}{2}$

6 **a** $-2 \leqslant f(x) \leqslant 18$
**b** 0
**c**




**d** $x = \dfrac{7 \pm \sqrt{5}}{2}$

# 3. Coordinate Geometry

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **3**<br><br>**Coordinate geometry in the $(x,y)$ plane** | 3.1 | **Understand and use the equation of a straight line, including the forms**<br>$y - y_1 = m(x - x_1)$ **and**<br>$ax + by + c = 0$;<br><br>**Gradient conditions for two straight lines to be parallel or perpendicular.**<br><br>**Be able to use straight line models in a variety of contexts.** | **To include the equation of a line through two given points, and the equation of a line parallel (or perpendicular) to a given line through a given point.**<br><br>$m' = m$ **for parallel lines and** $m' = -\frac{1}{m}$ **for perpendicular lines**<br><br>**For example, the line for converting degrees Celsius to degrees Fahrenheit, distance against time for constant speed, etc.** |
| | 3.2 | **Understand and use the coordinate geometry of the circle including using the equation of a circle in the form** $(x - a)^2 + (y - b)^2 = r^2$<br><br>**Completing the square to find the centre and radius of a circle; use of the following properties:**<br>• the angle in a semicircle is a right angle<br>• the perpendicular from the centre to a chord bisects the chord<br>• the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point. | **Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.**<br><br>**Students should also be familiar with the equation** $x^2 + y^2 + 2fx + 2gy + c = 0$<br><br>**Students should be able to find the equation of a circumcircle of a triangle with given vertices using these properties.**<br><br>**Students should be able to find the equation of a tangent at a specified point, using the perpendicular property of tangent and radius.** |

# 3. Coordinate Geometry

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **3**<br><br>**Coordinate geometry in the $(x, y)$ plane**<br><br>*continued* | 3.3 | Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms. | For example: $x = 3\cos t,\ y = 3\sin t$ describes a circle centre $O$ radius 3<br>$x = 2 + 5\cos t,\ y = -4 + 5\sin t$ describes a circle centre $(2, -4)$ with radius 5<br><br>$x = 5t,\ y = \frac{5}{t}$ describes the curve $xy = 25$<br><br>(or $y = \frac{25}{x}$)<br><br>$x = 5t, y = 3t^2$ describes the quadratic curve $25y = 3x^2$ and other familiar curves covered in the specification.<br><br>Students should pay particular attention to the domain of the parameter $t$, as a specific section of a curve may be described. |
| | 3.4 | Use parametric equations in modelling in a variety of contexts. | A shape may be modelled using parametric equations or students may be asked to find parametric equations for a motion. For example, an object moves with constant velocity from $(1, 8)$ at $t = 0$ to $(6, 20)$ at $t = 5$. This may also be tested in Paper 3, section 7 (kinematics). |

# 3. Coordinate Geometry

E/P 1 $A$ is the point $(-1, 5)$. Let $(x, y)$ be any point on the line $y = 3x$.

**a** Write an equation in terms of $x$ for the distance between $(x, y)$ and $A(-1, 5)$. **(3 marks)**

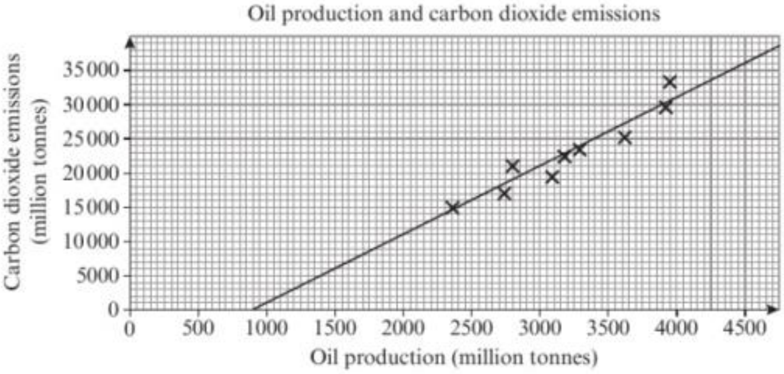
**b** Find the coordinates of the two points, $B$ and $C$, on the line $y = 3x$ which are a distance of $\sqrt{74}$ from $(-1, 5)$. **(3 marks)**

**c** Find the equation of the line $l_1$ that is perpendicular to $y = 3x$ and goes through the point $(-1, 5)$. **(2 marks)**

**d** Find the coordinates of the point of intersection between $l_1$ and $y = 3x$. **(2 marks)**

**e** Find the area of triangle $ABC$. **(2 marks)**

E/P 2 The scatter graph shows the oil production $P$ and carbon dioxide emissions $C$ for various years since 1970. A line of best fit has been added to the scatter graph.




**a** Use two points on the line to calculate its gradient. **(1 mark)**

**b** Formulate a linear model linking oil production $P$ and carbon dioxide emissions $C$, giving the relationship in the form $C = aP + b$. **(2 marks)**

**c** Interpret the value of $a$ in your model. **(1 mark)**

**d** With reference to your value of $b$, comment on the validity of the model for small values of $P$. **(1 mark)**

E/P 3 The circle $C$ has a centre at $(6, 9)$ and a radius of $\sqrt{50}$.

The line $l_1$ with equation $x + y - 21 = 0$ intersects the circle at the points $P$ and $Q$.

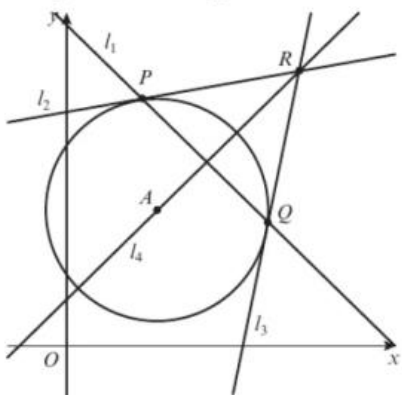
**a** Find the coordinates of the point $P$ and the point $Q$. **(5 marks)**

**b** Find the equations of $l_2$ and $l_3$, the tangents at the points $P$ and $Q$ respectively. **(4 marks)**

**c** Find the equation of $l_4$, the perpendicular bisector of the chord $PQ$. **(4 marks)**

**d** Show that the two tangents and the perpendicular bisector intersect and find the coordinates of $R$, the point of intersection. **(2 marks)**

**e** Calculate the area of the kite $APRQ$. **(3 marks)**



(P) 4 The line $y = -3x + 12$ meets the coordinate axes at $A$ and $B$.

**a** Find the coordinates of $A$ and $B$.

**b** Find the coordinates of the midpoint of $AB$.

**c** Find the equation of the circle that passes through $A$, $B$ and $O$, where $O$ is the origin.

(E/P) 5 A mountaineer's climb at time $t$ hours can be modelled with the following parametric equations

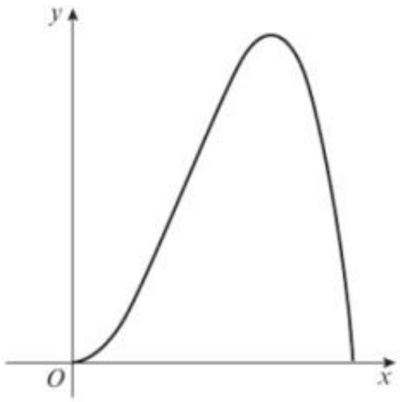
$$x = 300\sqrt{t}, \quad y = 244t(4 - t), \quad 0 < t < k$$

where $x$ represents the distance travelled horizontally in metres and $y$ represents the height above sea level in metres.

**a** Find the height of the peak and the time at which the mountaineer reaches it. **(3 marks)**

Given that the mountaineer completes her climb when she gets back to sea level,

**b** find the horizontal distance from the beginning of her climb to the end. **(2 marks)**




6 A BMX cyclist's position on a ramp at time $t$ seconds can be modelled with the parametric equations

$$x = 3(e^t - 1), \quad y = 10(t - 1)^2, \quad 0 \leqslant t \leqslant 1.3$$

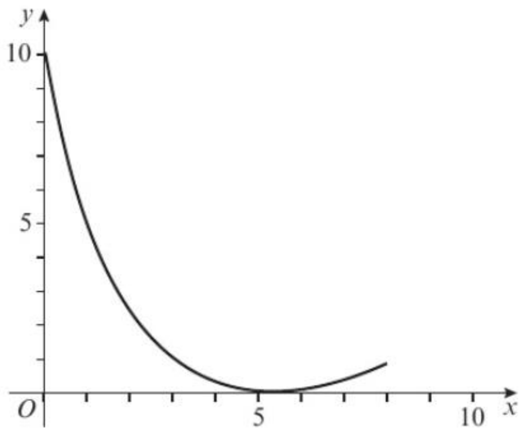
where $x$ is the horizontal distance travelled in metres and $y$ is the height above ground level in metres.

**a** Find the initial height of the cyclist.

**b** Find the time the cyclist is at her lowest height.

Given that after 1.3 seconds, the cyclist is at the end of the ramp,

**c** find the height at which the cyclist leaves the ramp.



# 3. Coordinate Geometry

## Answers

1 **a** $d = \sqrt{10x^2 - 28x + 26}$

**b** $B\left(-\frac{6}{5}, -\frac{18}{5}\right)$ and $C(4, 12)$

**c** $y = -\frac{1}{3}x + \frac{14}{3}$

**d** $\left(\frac{7}{5}, \frac{21}{5}\right)$

**e** 20.8

2 **a** gradient = 10

**b** $C = 10P - 9000$

**c** When the oil production increases by 1 million tonnes, the carbon dioxide emissions increase by 10 million tonnes.

**d** The model is not valid for small values of $P$, as it is not possible to have a negative amount of carbon dioxide emissions. It is always dangerous to extrapolate beyond the range on the model in this way.

3 **a** $P(5, 16)$ and $Q(13, 8)$

**b** $l_2$: $y = \frac{1}{7}x + \frac{107}{7}$ and $l_3$: $y = 7x - 83$

**c** $l_4$: $y = x + 3$

**d** All 3 equations have solution $x = \frac{43}{3}$, $y = \frac{52}{3}$ so $R(\frac{43}{3}, \frac{52}{3})$

**e** $\frac{200}{3}$

4 **a** $(4,0)$, $(0,12)$

**b** $(2,6)$

**c** $(x-2)^2 + (y-6)^2 = 40$

5 **a** 976 m, 2 hours **b** 600 m

6 **a** 10 m **b** 1 second **c** 0.9 m

# 4. Sequences And Series

| Topics | What students need to learn: | | |
|---|---|---|---|
| | | Content | Guidance |
| **4**<br>**Sequences and series** | 4.1 | **Understand and use the binomial expansion of $(a+bx)^n$ for positive integer $n$; the notations $n!$ and ${}^nC_r$ link to binomial probabilities.** | **Use of Pascal's triangle.**<br>**Relation between binomial coefficients.**<br>**Also be aware of alternative notations such as $\binom{n}{r}$ and ${}^nC_r$**<br>**Considered further in Paper 3 Section 4.1.** |
| | | Extend to any rational $n$, including its use for approximation; be aware that the expansion is valid for $\left\lvert\frac{bx}{a}\right\rvert < 1$ (proof not required) | May be used with the expansion of rational functions by decomposition into partial fractions<br>May be asked to comment on the range of validity. |
| **4**<br>**Sequences and series**<br>*continued* | 4.2 | Work with sequences including those given by a formula for the $n$ th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$;<br>increasing sequences; decreasing sequences; periodic sequences. | For example $u_n = \frac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integer $n$<br>$u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integer $n$<br>$u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2 |
| | 4.3 | Understand and use sigma notation for sums of series. | Knowledge that $\sum_{1}^{n} 1 = n$ is expected |
| | 4.4 | Understand and work with arithmetic sequences and series, including the formulae for $n$th term and the sum to $n$ terms | The proof of the sum formula for an arithmetic sequence should be known including the formula for the sum of the first $n$ natural numbers. |
| | 4.5 | Understand and work with geometric sequences and series, including the formulae for the $n$th term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $\lvert r\rvert < 1$; modulus notation | The proof of the sum formula should be known.<br>Given the sum of a series students should be able to use logs to find the value of $n$.<br>The sum to infinity may be expressed as $S_\infty$ |
| | 4.6 | Use sequences and series in modelling. | Examples could include amounts paid into saving schemes, increasing by the same amount (arithmetic) or by the same percentage (geometric) or could include other series defined by a formula or a relation. |

# 4. Sequences And Series

E 1 **a** Expand $(1 + 2x)^{12}$ in ascending powers of $x$ up to and including the term in $x^3$, simplifying each coefficient. **(4 marks)**

**b** By substituting a suitable value for $x$, which must be stated, into your answer to part **a**, calculate an approximate value of $1.02^{12}$. **(3 marks)**

**c** Use your calculator, writing down all the digits in your display, to find a more exact value of $1.02^{12}$. **(1 mark)**

**d** Calculate, to 3 significant figures, the percentage error of the approximation found in part **b**. **(1 mark)**

E/P 2 **a** Find the first three terms, in ascending powers of $x$ of the binomial expansion of $(2 + px)^7$, where $p$ is a constant. **(2 marks)**

The first 3 terms are 128, $2240x$ and $qx^2$, where $q$ is a constant.

**b** Find the value of $p$ and the value of $q$. **(4 marks)**

E/P 3 The fourth term of an arithmetic series is $3k$, where $k$ is a constant, and the sum of the first six terms of the series is $7k + 9$.

**a** Show that the first term of the series is $9 - 8k$. **(3 marks)**

**b** Find an expression for the common difference of the series in terms of $k$. **(2 marks)**

Given that the seventh term of the series is 12, calculate:

**c** the value of $k$ **(2 marks)**

**d** the sum of the first 20 terms of the series. **(2 marks)**

E/P 4 The adult population of a town is 25 000 at the beginning of 2012.
A model predicts that the adult population of the town will increase by 2% each year, forming a geometric sequence.

**a** Show that the predicted population at the beginning of 2014 is 26 010. **(1 mark)**

The model predicts that after $n$ years, the population will first exceed 50 000.

**b** Show that $n > \dfrac{\log 2}{\log 1.02}$ **(3 marks)**

**c** Find the year in which the population first exceeds 50 000. **(2 marks)**

**d** Every member of the adult population is modelled to visit the doctor once per year. Calculate the number of appointments the doctor has from the beginning of 2012 to the end of 2019. **(4 marks)**

**e** Give a reason why this model for doctors' appointments may not be appropriate. **(1 mark)**

E/P 5 **a** Use the binomial theorem to expand $(4+x)^{-\frac{1}{2}}$, $|x|<4$, in ascending powers of $x$, up to and including the $x^3$ term, giving each answer as a simplified fraction. **(5 marks)**

**b** Use your expansion, together with a suitable value of $x$, to obtain an approximation to $\frac{\sqrt{2}}{2}$. Give your answer to 4 decimal places. **(3 marks)**

E/P 6 $\mathrm{f}(x) = \dfrac{12x+5}{(1+4x)^2}$, $|x| < \dfrac{1}{4}$

For $x \neq -\dfrac{1}{4}$, $\dfrac{12x+5}{(1+4x)^2} = \dfrac{A}{1+4x} + \dfrac{B}{(1+4x)^2}$, where $A$ and $B$ are constants.

**a** Find the values of $A$ and $B$. **(3 marks)**

**b** Hence, or otherwise, find the series expansion of $\mathrm{f}(x)$, in ascending powers of $x$, up to and including the term $x^2$, simplifying each term. **(6 marks)**

# 4. Sequences And Series

## Answers

1 **a** $1 + 24x + 264x^2 + 1760x^3$ **b** 1.268 16
**c** 1.268 241 795 **d** 0.006 45% (3 sf)

2 **a** $128 + 448px + 672p^2x^2$
**b** $p = 5$, $q = 16\,800$

3 **a** $a + 3d = 3k$, $3(2a + 5d) = 7k + 9 \Rightarrow$
$6a + 15d = 7k + 9$
$6a + 15\left(\frac{3k - a}{3}\right) = 7k + 9$
$6a + 15k - 5a = 7k + 9 \Rightarrow a = 9 - 8k$
**b** $\frac{11k - 9}{3}$ **c** 1.5 **d** 415

4 **a** $25000 \times 1.02^2 = 26010$
**b** $25000 \times 1.02^n > 50000$
$1.02^n > 2 \Rightarrow n \log 1.02 > \log 2 \Rightarrow n > \frac{\log 2}{\log 1.02}$
**c** 2047
**d** 214574
**e** People may visit the doctor more frequently than once a year, some may not visit at all, depends on state of health

5 **a** $\frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \frac{5x^3}{2048}$ **b** 0.6914

6 **a** $A = 3$ and $B = 2$ **b** $5 - 28x + 144x^2$

# 5. Trigonometry

| Topics | What students need to learn: | | |
|---|---|---|---|
| | **Content** | | **Guidance** |
| **5**<br><br>**Trigonometry** | 5.1 | **Understand and use the definitions of sine, cosine and tangent for all arguments;**<br><br>**the sine and cosine rules;**<br><br>**the area of a triangle in the form $\frac{1}{2}ab\sin C$**<br><br>Work with radian measure, including use for arc length and area of sector. | **Use of $x$ and $y$ coordinates of points on the unit circle to give cosine and sine respectively,**<br><br>**including the ambiguous case of the sine rule.**<br><br>Use of the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for arc lengths and areas of sectors of a circle. |
| | 5.2 | Understand and use the standard small angle approximations of sine, cosine and tangent<br>$\sin\theta \approx \theta$,<br>$\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$<br>Where $\theta$ is in radians. | Students should be able to approximate, e.g. $\frac{\cos 3x - 1}{x\sin 4x}$ when $x$ is small, to $-\frac{9}{8}$ |
| | 5.3 | **Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.**<br><br>Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof. | **Knowledge of graphs of curves with equations such as $y = \sin x$, $y = \cos(x + 30°)$, $y = \tan 2x$ is expected.** |
| | 5.4 | Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains. | Angles measured in both degrees and radians. |

# 5. Trigonometry

| Topics | What students need to learn: | | |
|---|---|---|---|
| | | Content | Guidance |
| **5**<br>**Trigonometry**<br>*continued* | 5.5 | **Understand and use** $\tan\theta = \frac{\sin\theta}{\cos\theta}$<br>**Understand and use** $\sin^2\theta + \cos^2\theta = 1$<br>$\sec^2\theta = 1 + \tan^2\theta$ and<br>$\text{cosec}^2\theta = 1 + \cot^2\theta$ | **These identities may be used to solve trigonometric equations and angles may be in degrees** or radians. **They may also be used to prove further identities.** |
| | 5.6 | Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, and $\tan(A \pm B)$,<br>understand geometrical proofs of these formulae.<br>Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$ | To include application to half angles.<br>Knowledge of the $\tan(\frac{1}{2}\theta)$ formulae will *not* be required.<br>Students should be able to solve equations such as $a\cos\theta + b\sin\theta = c$ in a given interval. |
| | 5.7 | **Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.** | **Students should be able to solve equations such as**<br>$\sin(x + 70°) = 0.5$ for $0 < x < 360°$,<br>$3 + 5\cos 2x = 1$ for $-180° < x < 180°$<br>$6\cos^2 x + \sin x - 5 = 0, 0 \leq x < 360°$<br>These may be in degrees or radians and this will be specified in the question. |
| | 5.8 | Construct proofs involving trigonometric functions and identities. | Students need to prove identities such as $\cos x\cos 2x + \sin x\sin 2x \equiv \cos x$. |
| | 5.9 | Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces. | Problems could involve (for example) wave motion, the height of a point on a vertical circular wheel, or the hours of sunlight throughout the year. Angles may be measured in degrees or in radians. |

# 5. Trigonometry

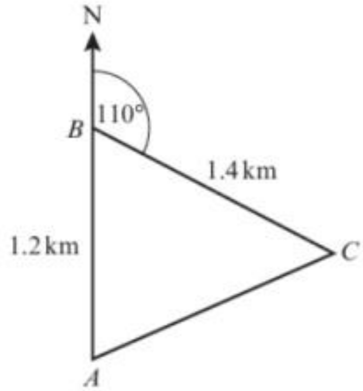
E/P 1 A park is in the shape of a triangle $ABC$ as shown.

A park keeper walks due north from his hut at $A$ until he reaches point $B$. He then walks on a bearing of 110° to point $C$.

**a** Find how far he is from his hut when at point $C$.
Give your answer in km to 3 s.f. **(3 marks)**

**b** Work out the bearing of the hut from point $C$.
Give your answer to the nearest degree. **(3 marks)**

**c** Work out the area of the park. **(3 marks)**




E 2 Find, in degrees, the values of $\theta$ in the interval $0 \leqslant \theta \leqslant 360°$ for which
$2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$
Give your answers to 1 decimal place, where appropriate. **(6 marks)**

E 3 **a** Express $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right)$ as a single trigonometric function. **(1 mark)**

**b** Hence solve $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right) = 1$ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers to 3 significant figures. **(3 marks)**

E/P 4 **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$. **(3 marks)**

**b** Deduce the values of:

**i** $\sec x$ **ii** $\tan x$ **(3 marks)**

**c** Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$. **(3 marks)**

E/P 5 **a** Prove that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$. **(4 marks)**

**b** Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2\cos 2\theta$. **(1 mark)**

**c** Using the result in part **a**, or otherwise, find the two other solutions, $0 < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2\cos 2\theta$. **(3 marks)**

E/P 6 **a** Express $1.4\sin\theta - 5.6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R$ and $\alpha$ are constants, $R > 0$ and $0 < \alpha < 90°$. Round $R$ and $\alpha$ to 3 decimal places. **(4 marks)**

**b** Hence find the maximum value of $1.4\sin\theta - 5.6\cos\theta$ and the smallest positive value of $\theta$ for which this maximum occurs. **(3 marks)**

The length of daylight, $d(t)$ at a location in northern Scotland can be modelled using the equation

$$d(t) = 12 - 5.6\cos\left(\frac{360t}{365}\right)^\circ + 1.4\sin\left(\frac{360t}{365}\right)^\circ$$

where $t$ is the numbers of days into the year.

**c** Calculate the minimum number of daylight hours in northern Scotland as given by this model. **(2 marks)**

**d** Find the value of $t$ when this minimum number of daylight hours occurs. **(1 mark)**

# 5. Trigonometry

## Answers

1 **a** 1.50 km **b** 241° **c** 0.789 $km^2$

2 131.8°, 228.2°

3 **a** $3\sin\theta$ **b** 0.340, 2.80

4 **a** $-\frac{1}{3}$ **b** **i** $-\frac{5}{3}$, **ii** $-\frac{4}{3}$ **c** 126.9°

5 **a** Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
**b** $\sin 360° = 0$, $2 - 2\cos(360°) = 2 - 2 = 0$
**c** 26.6°, 206.6°

6 **a** $R = 5.772$, $\alpha = 75.964°$ **b** 5.772 when $\theta = 166.0°$
**c** 6.228 hours **d** 350.8 days

# 6. Exponentials and Logarithms

| Topics | | What students need to learn: | |
|---|---|---|---|
| | | Content | Guidance |
| **6**<br><br>**Exponentials and logarithms** | 6.1 | **Know and use the function $a^x$ and its graph, where $a$ is positive.**<br><br>**Know and use the function $e^x$ and its graph.** | **Understand the difference in shape between $a < 1$ and $a > 1$**<br><br>To include the graph of $y = e^{ax+b} + c$ |
| | 6.2 | **Know that the gradient of $e^{kx}$ is equal to $ke^{kx}$ and hence understand why the exponential model is suitable in many applications.** | **Realise that when the rate of change is proportional to the $y$ value, an exponential model should be used.** |
| | 6.3 | **Know and use the definition of $\log_a x$ as the inverse of $a^x$, where $a$ is positive and $x > 0$.**<br><br>**Know and use the function $\ln x$ and its graph.**<br><br>**Know and use $\ln x$ as the inverse function of $e^x$** | $a \neq 1$<br><br>**Solution of equations of the form $e^{ax+b} = p$ and $\ln(ax + b) = q$ is expected.** |
| | 6.4 | **Understand and use the laws of logarithms:**<br><br>$\log_a x + \log_a y = \log_a (xy)$<br><br>$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$<br><br>$k \le \log_a x = \log_a x^k$<br><br>**(including, for example, $k = -1$ and $k = -\frac{1}{2}$)** | **Includes $\log_a a = 1$** |
| | 6.5 | **Solve equations of the form $a^x = b$** | **Students may use the change of base formula. Questions may be of the form, e.g. $2^{3x-1} = 3$** |
| | 6.6 | **Use logarithmic graphs to estimate parameters in relationships of the form**<br><br>**$y = ax^n$ and $y = kb^x$, given data for $x$ and $y$** | **Plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log a$ and the gradient is $n$**<br><br>**Plot $\log y$ against $x$ and obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$** |
| **6**<br><br>**Exponentials and logarithms**<br><br>*continued* | 6.7 | **Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.** | **Students may be asked to find the constants used in a model.**<br><br>**They need to be familiar with terms such as initial, meaning when $t = 0$.**<br><br>**They may need to explore the behaviour for large values of $t$ or to consider whether the range of values predicted is appropriate.**<br><br>**Consideration of a second improved model may be required.** |

# 6. Exponentials and Logarithms

(P) 1 The points $P$ and $Q$ lie on the curve with equation $y = e^{\frac{1}{2}x}$.
The $x$-coordinates of $P$ and $Q$ are $\ln 4$ and $\ln 16$ respectively.

**a** Find an equation for the line $PQ$.

**b** Show that this line passes through the origin $O$.

**c** Calculate the length, to 3 significant figures, of the line segment $PQ$.

(P) 2 'he total number of views (in millions) $V$ of a viral video in $x$ days is modelled by

$$V = e^{0.4x} - 1$$

**a** Find the total number of views after 5 days, giving your answer to 2 significant figures.

**b** Find $\frac{dV}{dx}$.

(P) 3 The moment magnitude scale is used by seismologists to express the sizes of earthquakes. The scale is calculated using the formula

$$M = \tfrac{2}{3}\log_{10}(S) - 10.7$$

where $S$ is the seismic moment in dyne cm.

**a** Find the magnitude of an earthquake with a seismic moment of $2.24 \times 10^{22}$ dyne cm.

**b** Find the seismic moment of an earthquake with
**i** magnitude 6 **ii** magnitude 7

**c** Using your answers to part **b** or otherwise, show that an earthquake of magnitude 7 is approximately 32 times as powerful as an earthquake of magnitude 6.

(E/P) 4 A student is asked to solve the equation

$$\log_2 x - \tfrac{1}{2}\log_2(x + 1) = 1$$

The student's attempt is shown

$$\log_2 x - \log_2\sqrt{x + 1} = 1$$
$$x - \sqrt{x + 1} = 2^1$$
$$x - 2 = \sqrt{x + 1}$$
$$(x - 2)^2 = x + 1$$
$$x^2 - 5x + 3 = 0$$
$$x = \frac{5 + \sqrt{13}}{2} \quad x = \frac{5 - \sqrt{13}}{2}$$

**a** Identify the error made by the student. **(1 mark)**

**b** Solve the equation correctly. **(3 marks)**

5 The population, $P$, of a colony of endangered Caledonian owlet-nightjars can be modelled by the equation $P = ab^t$ where $a$ and $b$ are constants and $t$ is the time, in months, since the population was first recorded.

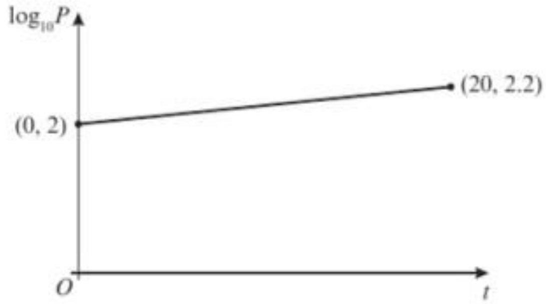



**Figure 2**

The line $l$ shown in figure 2 shows the relationship between $t$ and $\log_{10}P$ for the population over a period of 20 years.

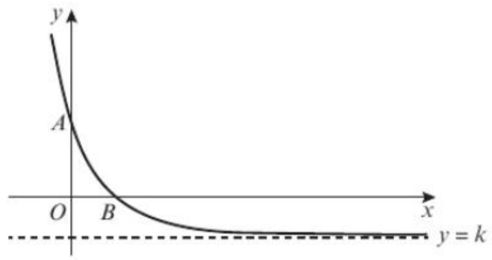
**a** Write down an equation of line $l$. **(3)**

**b** Work out the value of $a$ and interpret this value in the context of the model. **(3)**

**c** Work out the value of $b$, giving your answer correct to 3 decimal places. **(2)**

**d** Find the population predicted by the model when $t = 30$. **(1)**

E 6 The graph of the function $f(x) = 3e^{-x} - 1$, $x \in \mathbb{R}$, has an asymptote $y = k$, and crosses the $x$ and $y$ axes at $A$ and $B$ respectively, as shown in the diagram.




**a** Write down the value of $k$ and the $y$-coordinate of $A$. **(2 marks)**

**b** Find the exact value of the $x$-coordinate of $B$, giving your answer as simply as possible. **(2 marks)**

# 6. Exponentials and Logarithms

## Answers

1 **a** $y = \left(\frac{2}{\ln 4}\right)x$

**b** (0, 0) satisfies the equation of the line.

**c** 2.43

2 **a** 6.4 million views

**b** $\frac{dV}{dx} = 0.4e^{0.4x}$

**c** $9.42 \times 10^{16}$ new views per day

**d** This is too big, so the model is not valid after 100 days

3 **a** 4.2

**b** **i** $1.12 \times 10^{25}$ dyne cm

**ii** $3.55 \times 10^{26}$ dyne cm

**c** divide **b ii** by **b i**

4 **a** They exponentiated the two terms on LHS separately rather than combining them first.

**b** $x = 2 + 2\sqrt{2}$

5 **a** $\log_{10} P = 0.01t + 2$

**b** 100, initial population

**c** 1.023

**d** Accept answers from 195 to 200

6 **a** $k = -1$, $A(0, 2)$

**b** $\ln 3$

# 7. Differentiation

| Topics | | What students need to learn: | |
|---|---|---|---|
| | | **Content** | **Guidance** |
| **7**<br>**Differentiation** | 7.1 | **Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point $(x, y)$; the gradient of the tangent as a limit; interpretation as a rate of change** | **Know that $\frac{dy}{dx}$ is the rate of change of $y$ with respect to $x$.**<br>**The notation $f'(x)$ may be used for the first derivative and $f''(x)$ may be used for the second derivative.** |
| | | **sketching the gradient function for a given curve** | **Given for example the graph of $y = f(x)$, sketch the graph of $y = f'(x)$ using given axes and scale. This could relate speed and acceleration for example.** |
| | | **second derivatives** | |
| | | **differentiation from first principles for small positive integer powers of $x$ and for $\sin x$ and $\cos x$** | **For example, students should be able to use, for $n = 2$ and $n = 3$, the gradient expression**<br>$\lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$<br>**Students may use $\delta x$ or $h$** |
| **7**<br>**Differentiation**<br>*continued* | 7.1<br>*cont.* | **Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection. | Use the condition $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum for points where $f'(x) = 0$<br>Know that at an inflection point $f''(x)$ changes sign.<br>Consider cases where $f''(x) = 0$ and $f'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n, n > 2$) |
| | 7.2 | **Differentiate $x^n$, for rational values of $n$, and related constant multiples, sums and differences.** | **For example, the ability to differentiate expressions such as**<br>$(2x+5)(x-1)$ **and** $\frac{x^2+3x-5}{4x^{\frac{1}{2}}}, x > 0,$<br>**is expected.** |
| | | Differentiate $e^{kx}$ and $a^{kx}$, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. | Knowledge and use of the result $\frac{d}{dx}(a^{kx}) = ka^{kx}\ln a$ is expected. |
| | | Understand and use the derivative of $\ln x$ | |
| | 7.3 | **Apply differentiation to find gradients, tangents and normals** | **Use of differentiation to find equations of tangents and normals at specific points on a curve.** |
| | | **maxima and minima and stationary points.**<br>points on inflection | **To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.** |
| | | **Identify where functions are increasing or decreasing.** | **To include applications to curve sketching.** |
| | 7.4 | Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions. | Differentiation of $\operatorname{cosec} x$, $\cot x$ and $\sec x$ and differentiation of $\arcsin x$, $\arccos x$, and $\arctan x$ are required. Skill will be expected in the differentiation of functions generated from standard forms using products, quotients and composition, such as $2x^4 \sin x$, $\frac{e^{3x}}{x}$, $\cos^2 x$ and $\tan^2 2x$. |

# 7. Differentiation

1 A curve has equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. Find the equations of the tangent and the normal to the curve at the point where $x = 2$.

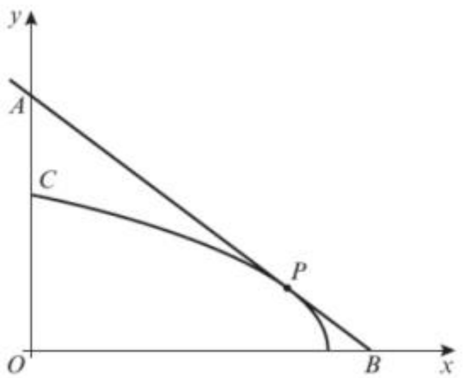
P 2 The total surface area, $A$ cm$^2$, of a cylinder with a fixed volume of 1000 cm$^3$ is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where $x$ cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$

E/P 3 Given that a curve has equation $y = \cos^2 x + \sin x$, $0 < x < 2\pi$, find the coordinates of the stationary points of the curve. **(6 marks)**

E/P 4 The diagram shows the curve $C$ with parametric equations

$$x = a\sin^2 t, \quad y = a\cos t, \quad 0 \leqslant t \leqslant \tfrac{1}{2}\pi$$

where $a$ is a positive constant. The point $P$ lies on $C$ and has coordinates $\left(\frac{3}{4}a, \frac{1}{2}a\right)$.




**a** Find $\frac{dy}{dx}$, giving your answer in terms of $t$. **(4 marks)**

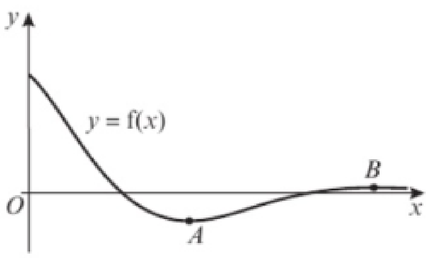
**b** Find an equation of the tangent to $C$ at $P$. **(4 marks)**

The tangent to $C$ at $P$ cuts the coordinate axes at points $A$ and $B$.

**c** Show that the triangle $AOB$ has area $ka^2$ where $k$ is a constant to be found. **(2 marks)**

E/P 5 A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. $A$ and $B$ are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that the straight line passing through $A$ and $B$ has equation $x + 2y = 0$. **(6 marks)**

E/P 6 The curve C with equation $y = f(x)$ is shown in the diagram, where $f(x) = \frac{\cos 2x}{e^x}$, $0 \leqslant x \leqslant \pi$




The curve has a local minimum at $A$ and a local maximum at $B$.

**a** Show that the $x$-coordinates of $A$ and $B$ satisfy the equation $\tan 2x = -0.5$ and hence find the coordinates of $A$ and $B$. **(6 marks)**

**b** Using your answer to part **a**, find the coordinates of the maximum and minimum turning points on the curve with equation $y = 2 + 4f(x - 4)$. **(3 marks)**

**c** Determine the values of $x$ for which $f(x)$ is concave. **(5 marks)**

# 7. Differentiation

## Answers

1 $y = 9x - 4$ and $9y + x = 128$

3 $\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right), \left(\frac{3\pi}{2}, -1\right)$

4 **a** $-\frac{1}{2}\sec t$ **b** $4y + 4x = 5a$

**c** Tangent crosses the $x$-axis at $x = \frac{5}{4}a$, and crosses the $y$-axis at $y = \frac{5}{4}a$.

So area $AOB = \frac{1}{2}\left(\frac{5}{4}a\right)^2 = \frac{25}{32}a^2$, $k = \frac{25}{32}$

5 $14x + 48y + 48x\frac{dy}{dx} - 14y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-7x - 24y}{24x - 7y}$

So $\frac{-7x - 24y}{24x - 7y} = \frac{2}{11} \Rightarrow -77x - 264y$

$= 48x - 14y \Rightarrow x + 2y = 0$

6 **a** $f'(x) = -\frac{2\sin 2x + \cos 2x}{e^x}$

$f'(x) = 0 \Leftrightarrow 2\sin 2x + \cos 2x = 0 \Leftrightarrow \tan 2x = -0.5$

$A(1.34, -0.234)$, $B(2.91, 0.0487)$

**b** Maximum (6.91, 2.19); minimum (5.34, 1.06) to 3 s.f.

**c** $0 < x \leqslant 0.322$, $1.89 \leqslant x < \pi$

# 8. Integration

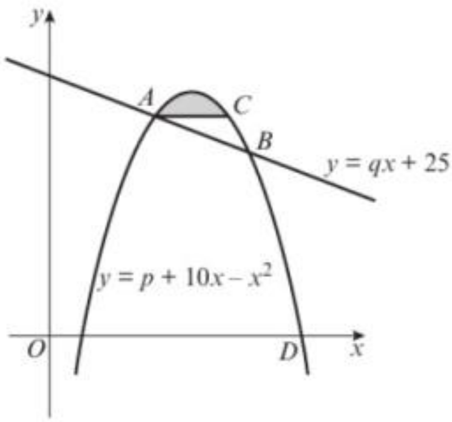
| Topics | What students need to learn: | | |
|---|---|---|---|
| | | Content | Guidance |
| **8**<br>**Integration** | 8.1 | **Know and use the Fundamental Theorem of Calculus** | **Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.** |
| | 8.2 | **Integrate $x^n$ (excluding $n = -1$) and related sums, differences and constant multiples.**<br><br>Integrate $e^{kx}$, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples. | **For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected.**<br>**Given $f'(x)$ and a point on the curve, Students should be able to find an equation of the curve in the form $y = f(x)$.**<br><br>To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, $e^{5x}$, $\frac{1}{2x}$.<br><br>Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$. |
| | 8.3 | **Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves | Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically.<br><br>For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$<br><br>Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$. |
| | 8.4 | Understand and use integration as the limit of a sum. | Recognise $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$ |

# 8. Integration

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **8** **Integration** *continued* | 8.5 | Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively<br>(Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.) | Students should recognise integrals of the form $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$.<br>The integral $\int \ln x \, dx$ is required<br>Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae. |
| | 8.6 | Integrate using partial fractions that are linear in the denominator. | Integration of rational expressions such as those arising from partial fractions, e.g. $\frac{2}{3x+5}$<br>Note that the integration of other rational expressions, such as $\frac{x}{x^2+5}$ and $\frac{2}{(2x-1)^4}$ is also required (see previous paragraph). |
| | 8.7 | Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions<br>(Separation of variables may require factorisation involving a common factor.) | Students may be asked to sketch members of the family of solution curves. |
| | 8.8 | Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics. | The validity of the solution for large values should be considered. |

# 8. Integration

E/P 1 The diagram shows part of the curve with equation $y = p + 10x - x^2$, where $p$ is a constant, and part of the line $l$ with equation $y = qx + 25$, where $q$ is a constant. The line $l$ cuts the curve at the points $A$ and $B$. The $x$-coordinates of $A$ and $B$ are 4 and 8 respectively. The line through $A$ parallel to the $x$-axis intersects the curve again at the point $C$.




**a** Show that $p = -7$ and calculate the value of $q$. **(3 marks)**

**b** Calculate the coordinates of $C$. **(2 marks)**

**c** The shaded region in the diagram is bounded by the curve and the line segment $AC$. Using integration and showing all your working, calculate the area of the shaded region. **(6 marks)**

E 2 Using the substitution $t^2 = x + 1$, where $x > -1$,

**a** find $\int \frac{x}{\sqrt{x+1}}\,dx$. **(5 marks)**

**b** Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. **(2 marks)**

E 3 **a** Use integration by parts to find $\int x \sin 8x\,dx$. **(4 marks)**

**b** Use your answer to part **a** to find $\int x^2 \cos 8x\,dx$. **(4 marks)**

E/P 4 $f(x) = \frac{5x^2 - 8x + 1}{2x(x-1)^2}$

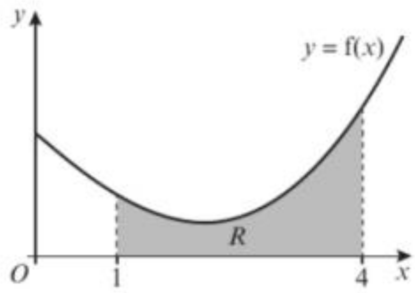
**a** Given that $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, find the values of the constants $A$, $B$ and $C$. **(4 marks)**

**b** Hence find $\int f(x)\,dx$. **(4 marks)**

**c** Hence show that $\int_4^9 f(x)\,dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$ **(4 marks)**

(E/P) 5 The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = \frac{1}{5}x^2 \ln x - x + 2$, $x > 0$.

The region $R$, shown in the diagram, is bounded by the curve, the $x$-axis and the lines with equations $x = 1$ and $x = 4$.




The table below shows the corresponding values of $x$ and $y$ with the $y$ values given to 4 decimal places as appropriate.

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 1 | 0.6825 | 0.5545 | 0.6454 | | 1.5693 | 2.4361 |

**a** Complete the table with the missing value of $y$. **(1 mark)**

**b** Use the trapezium rule, with all the values of $y$ in the table, to obtain an estimate for the area of $R$, giving your answer to 3 decimal places. **(3 marks)**

**c** Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of $R$. **(1 mark)**

**d** Show that the exact area of $R$ can be written in the form $\frac{a}{b} + \frac{c}{d} \ln e$, where $a$, $b$, $c$, $d$ and $e$ are integers. **(6 marks)**

**e** Find the percentage error in the answer in part **b**. **(2 marks)**

(E/P) 6 An oil spill is modelled as a circular disc with radius $r$ km and area $A$ km². The rate of increase of the area of the oil spill, in km²/day at time $t$ days after it occurs is modelled as:

$$\frac{dA}{dt} = k \sin\left(\frac{t}{3\pi}\right), \ 0 \leqslant t \leqslant 12$$

**a** Show that $\frac{dr}{dt} = \frac{k}{2\pi r} \sin\left(\frac{t}{3\pi}\right)$ **(2 marks)**

Given that the radius of the spill at time $t = 0$ is 1 km, and the radius of the spill at time $t = \pi^2$ is 2 km:

**b** find an expression for $r^2$ in terms of $t$ **(7 marks)**

**c** find the time, in days and hours to the nearest hour, after which the radius of the spill is 1.5 km. **(3 marks)**

# 8. Integration

## Answers

1 **a** $q = -2$ **b** $C(6,17)$ **c** $1\frac{1}{3}$

2 **a** $\frac{2}{3}(x-2)\sqrt{x+1}+c$ **b** $\frac{8}{3}$

3 **a** $-\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x + c$

**b** $\frac{1}{8}x^2\sin 8x + \frac{1}{32}x\cos 8x - \frac{1}{256}\sin 8x + c$

4 **a** $A = \frac{1}{2}, B = 2, C = -1$

**b** $\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + c$

**c** $\int_4^9 f(x)\,dx = \left[\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1}\right]_4^9$

$$= \left(\frac{1}{2}\ln 9 + 2\ln 8 + \frac{1}{8}\right) - \left(\frac{1}{2}\ln 4 + 2\ln 3 + \frac{1}{3}\right)$$

$$= \left(\ln 3 + \ln 64 + \frac{1}{8}\right) - \left(\ln 2 + \ln 9 + \frac{1}{3}\right)$$

$$= \ln\left(\frac{3\times 64}{2\times 9}\right) - \frac{5}{24} = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$$

5 **a** 0.9775 **b** 3.074

**c** Use more values, use smaller intervals. The lines would then more closely follow the curve.

**d** $\int_1^4 \left(\frac{1}{5}x^2\right)\ln x - x + 2\,dx$

$$= \left[\frac{1}{15}x^3\ln x - \frac{1}{45}x^3 - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \left(\frac{64}{15}\ln 4 - \frac{64}{45}\right) - \left(-\frac{1}{45} - \frac{1}{2} + 2\right) = \frac{-29}{10} + \frac{64}{15}\ln 4$$

**e** 2.0%

6 **a** $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times k\sin\left(\frac{t}{3\pi}\right) = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$$

**b** $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$ **c** 6 days, 5 hours

# 9. Numerical Methods

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **9**<br>**Numerical methods** | 9.1 | Locate roots of $\mathrm{f}(x) = 0$ by considering changes of sign of $\mathrm{f}(x)$ in an interval of $x$ on which $\mathrm{f}(x)$ is sufficiently well behaved.<br><br>Understand how change of sign methods can fail. | Students should know that sign change is appropriate for continuous functions in a small interval.<br><br>When the interval is too large sign may not change as there may be an even number of roots.<br><br>If the function is not continuous, sign may change but there may be an asymptote (not a root). |
| | 9.2 | Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams. | Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy.<br><br>Use an iteration of the form $x_{n+1} = \mathrm{f}(x_n)$ to find a root of the equation $x = \mathrm{f}(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams. |
| | 9.2 | Solve equations using the Newton-Raphson method and other recurrence relations of the form<br>$x_{n+1} = \mathrm{g}(x_n)$<br><br>Understand how such methods can fail. | For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small. |
| | 9.3 | Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between. | For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$<br><br>using the values of $\sqrt{(2x+1)}$ at $x = 0$, 0.25, 0.5, 0.75 and 1 and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate. |
| | 9.4 | Use numerical methods to solve problems in context. | Iterations may be suggested for the solution of equations not soluble by analytic means. |

# 9. Numerical Methods

E/P 1 $f(x) = x^3 - 6x - 2$

**a** Show that the equation $f(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$, and state the values of the integers $a$ and $b$. **(2 marks)**

$f(x) = 0$ has one positive root, $\alpha$.

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 2$ is used to find an approximate value for $\alpha$.

**b** Calculate the values of $x_1$, $x_2$, $x_3$ and $x_4$ to 4 decimal places. **(3 marks)**

**c** By choosing a suitable interval, show that $\alpha = 2.602$ is correct to 3 decimal places. **(3 marks)**

E/P 2 $g(x) = x^2 - 3x - 5$

**a** Show that the equation $g(x) = 0$ can be written as $x = \sqrt{3x + 5}$. **(1 mark)**

**b** Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{3x + 5}$. **(2 marks)**

**c** Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{3x_n + 5}$ converges to a root of $g(x)$ when $x_0 = 1$. **(1 mark)**

$g(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \frac{x_n^2 - 5}{3}$

**d** With reference to a diagram, explain why this iterative formula diverges when $x_0 = 7$. **(3 marks)**

E/P 3 $g(x) = x^3 - 7x^2 + 2x + 4$

**a** Find $g'(x)$. **(2 marks)**

A root $\alpha$ of the equation $g(x) = 0$ lies in the interval [6.5, 6.7].

**b** Taking 6.6 as a first approximation to $\alpha$, apply the Newton–Raphson process once to $g(x)$ to obtain a second approximation to $\alpha$. Give your answer to 3 decimal places. **(4 marks)**

**c** Given that $g(1) = 0$, find the exact value of the other two roots of $g(x)$. **(3 marks)**

**d** Calculate the percentage error of your answer in part **b**. **(2 marks)**

E/P 4 $f(x) = e^{0.8x} - \frac{1}{3 - 2x}$, $x \neq \frac{3}{2}$

**a** Show that the equation $f(x) = 0$ can be written as $x = 1.5 - 0.5e^{-0.8x}$. **(3 marks)**

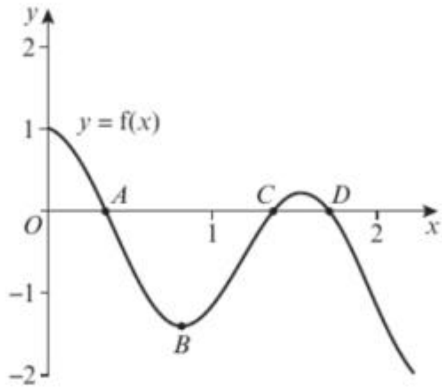
**b** Use the iterative formula $x_{n+1} = 1.5 - 0.5e^{-0.8x_n}$ with $x_0 = 1.3$ to obtain $x_1$, $x_2$ and $x_3$. Hence write down one root of $f(x) = 0$ correct to 3 decimal places. **(2 marks)**

**c** Show that the equation $f(x) = 0$ can be written in the form $x = p\ln(3 - 2x)$, stating the value of $p$. **(3 marks)**

**d** Use the iterative formula $x_{n+1} = p\ln(3 - 2x_n)$ with $x_0 = -2.6$ and the value of $p$ found in part **c** to obtain $x_1$, $x_2$ and $x_3$. Hence write down a second root of $f(x) = 0$ correct to 2 decimal places. **(2 marks)**

# 9. Numerical Methods

E/P 5 The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = \cos(4x) - \frac{1}{2}x$.




**a** Show that the curve has a root in the interval [1.3, 1.4]. **(2 marks)**

**b** Use differentiation to find the coordinates of point $B$. Write each coordinate correct to 3 decimal places. **(3 marks)**

**c** Using the iterative formula $x_{n+1} = \frac{1}{4}\arccos\left(\frac{1}{2}x_n\right)$, with $x_0 = 0.4$, find the values of $x_1$, $x_2$, $x_3$ and $x_4$. Give your answers to 4 decimal places. **(3 marks)**

**d** Using $x_0 = 1.7$ as a first approximation to the root at $D$, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to the root, giving your answer to 3 decimal places. **(4 marks)**

**e** By considering the change of sign of $f(x)$ over an appropriate interval, show that the answer to part **d** is accurate to 3 decimal places. **(2 marks)**

E/P 6 **a** On the same axes, sketch the graphs of $y = \frac{1}{x}$ and $y = x + 3$. **(2 marks)**

**b** Write down the number of roots of the equation $\frac{1}{x} = x + 3$. **(1 mark)**

**c** Show that the positive root of the equation $\frac{1}{x} = x + 3$ lies in the interval (0.30, 0.31). **(2 marks)**

**d** Show that the equation $\frac{1}{x} = x + 3$ may be written in the form $x^2 + 3x - 1 = 0$. **(2 marks)**

**e** Use the quadratic formula to find the positive root of the equation $x^2 + 3x - 1 = 0$ to 3 decimal places. **(2 marks)**

# 9. Numerical Methods

## Answers

1 **a** $x^3 - 6x - 2 = 0 \Rightarrow x^3 = 6x + 2$

$\Rightarrow x^2 = 6 + \frac{2}{x} \Rightarrow x = \pm\sqrt{6 + \frac{2}{x}}$; $a = 6$, $b = 2$

**b** $x_1 = 2.6458, x_2 = 2.5992, x_3 = 2.6018, x_4 = 2.6017$

**c** $f(2.6015) = (2.6015)^3 - 6(2.6015) - 2 = -0.0025... < 0$

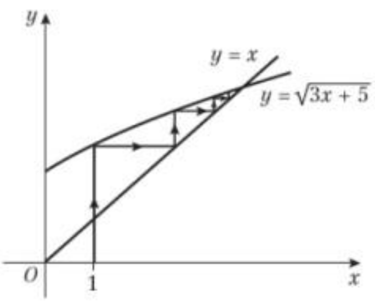
$f(2.6025) = (2.6025)^3 - 6(2.6025) - 2 = 0.0117 > 0$

There is a sign change in the interval $2.6015 < x < 2.6025$, so this implies there is a root in the interval.

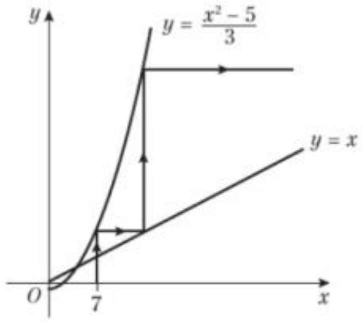
2 **a** $g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$

$\Rightarrow x^2 = 3x + 5 \Rightarrow x = \sqrt{3x + 5}$

**b, c**




**d**



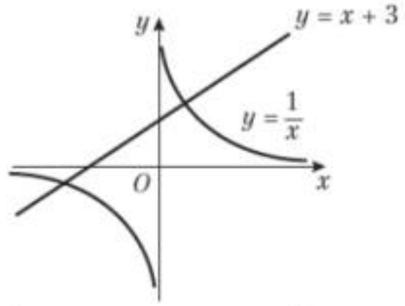
# 9. Numerical Methods

3 **a** $g'(x) = 3x^2 - 14x + 2$ **b** 6.606
**c** $(x - 1)(x^2 - 6x - 4) \Rightarrow x^2 - 6x - 4 = 0 \Rightarrow x = 3 \pm \sqrt{13}$
**d** 0.007%

4 **a** $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow (3 - 2x)e^{0.8x} - 1 = 0$
$\Rightarrow (3 - 2x)e^{0.8x} = 1 \Rightarrow 3 - 2x = e^{-0.8x}$
$\Rightarrow 3 - e^{-0.8x} = 2x \Rightarrow x = 1.5 - 0.5e^{-0.8x}$
**b** $x_1 = 1.32327\ldots$, $x_2 = 1.32653\ldots$, $x_3 = 1.32698\ldots$,
root = 1.327 (3 d.p.)
**c** $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow e^{0.8x} = \frac{1}{3 - 2x} \Rightarrow 3 - 2x = e^{-0.8x}$
$\Rightarrow -0.8x = \ln(3 - 2x) \Rightarrow x = -1.25\ln(3 - 2x)$
$p = -1.25$
**d** $x_1 = -2.6302$, $x_2 = -2.6393$, $x_3 = -2.6421$,
root = $-2.64$ (2 d.p.)

5 **a** $f(1.3) = -0.18148\ldots$, $f(1.4) = 0.07556\ldots$.. There is a sign change in the interval [1.3, 1.4], so there is a root in this interval.
**b** $(0.817, -1.401)$
**c** $x_1 = 0.3424$, $x_2 = 0.3497$, $x_3 = 0.3488$, $x_4 = 0.3489$
**d** $x_1 = 1.708$
**e** $f(1.7075) = 0.000435\ldots$., $f(1.7085) = -0.002151\ldots$.
There is a sign change in the interval [1.7075, 1.7085], so there is a root in this interval.

6 **a**




**b** 2

**c** $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$, let $f(x) = x + 3 - \frac{1}{x}$
$f(0.30) = -0.0333\ldots < 0$, $f(0.31) = 0.0841\ldots > 0$.
Sign change implies root.
**d** $\frac{1}{x} = x + 3 \Rightarrow 1 = x^2 + 3x \Rightarrow 0 = x^2 + 3x - 1$
**e** 0.303

# 10. Vectors

| Topics | What students need to learn: | | |
|---|---|---|---|
| | Content | | Guidance |
| **10** **Vectors** | 10.1 | **Use vectors in two dimensions** and in three dimensions | **Students should be familiar with column vectors and with the use of i and j unit vectors in two dimensions** and **i**, **j** and **k** unit vectors in three dimensions. |
| | 10.2 | **Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.** | **Students should be able to find a unit vector in the direction of *a*, and be familiar with the notation $\lvert a \rvert$.** |
| | 10.3 | **Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.** | **The triangle and parallelogram laws of addition.**<br><br>**Parallel vectors.** |
| | 10.4 | **Understand and use position vectors; calculate the distance between two points represented by position vectors.** | $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$<br><br>**The distance $d$ between two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by**<br>$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ |
| **10** **Vectors** | 10.5 | **Use vectors to solve problems in pure mathematics and in context, (including forces).** | **For example, finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners $A$, $B$ and $C$.**<br><br>**Or use of ratio theorem to find position vector of a point $C$ dividing $AB$ in a given ratio.**<br><br>**Contexts such as velocity, displacement, kinematics and forces will be covered in Paper 3, Sections 6.1, 7.3 and 8.1 – 8.4** |

# 10. Vectors

E/P 1 The resultant of the vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 2p\mathbf{i} - p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$. Find:

**a** the value of $p$ **(3 marks)**

**b** the resultant of vectors **a** and **b**. **(1 mark)**

E/P 2 Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are given by the vectors $\mathbf{F}_1 = (4\mathbf{i} - 5\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N. The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ acts in a direction which is parallel to the vector $(3\mathbf{i} - \mathbf{j})$

**a** Find the angle between **R** and the vector **i**. **(3 marks)**

**b** Show that $p + 3q = 11$. **(4 marks)**

**c** Given that $p = 2$, find the magnitude of **R**. **(2 marks)**

E 3 $P$ is the point $(-6, 2, 1)$, $Q$ is the point $(3, -2, 1)$ and $R$ is the point $(1, 3, -2)$.

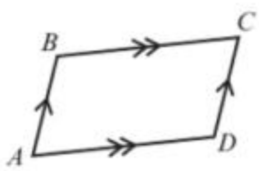
**a** Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$. **(3 marks)**

**b** Hence find the lengths of the sides of triangle $PQR$. **(6 marks)**

**c** Given that angle $QRP = 90°$ find the size of angle $PQR$. **(2 marks)**

E/P 4 The diagram shows the quadrilateral $ABCD$.

Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -2 \\ 11 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 15 \\ 8 \\ 5 \end{pmatrix}$, find the area of the quadrilateral. **(7 marks)**




P 5 $A$ is the point $(2, 3, -2)$, $B$ is the point $(0, -2, 1)$ and $C$ is the point $(4, -2, -5)$. When $A$ is reflected in the line $BC$ it is mapped to the point $D$.

**a** Work out the coordinates of the point $D$.

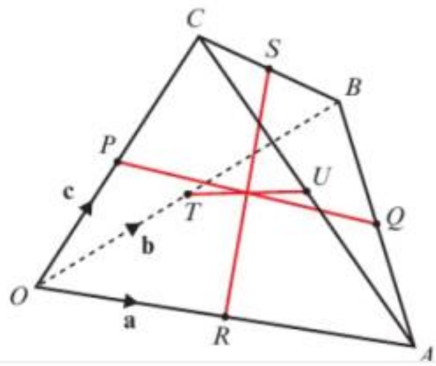
**b** Give the mathematical name for the shape $ABCD$.

**c** Work out the area of $ABCD$.

P 6 The diagram shows a tetrahedron $OABC$. **a**, **b** and **c** are the position vectors of $A$, $B$ and $C$ respectively.

$P$, $Q$, $R$, $S$, $T$ and $U$ are the midpoints of $OC$, $AB$, $OA$, $BC$, $OB$ and $AC$ respectively.

Prove that the line segments $PQ$, $RS$ and $TU$ meet at a point and bisect each other.



# 10. Vectors

## Answers

1 **a** $p = -1.5$ **b** $\mathbf{i} - 1.5\mathbf{j}$

2 **a** 18.4° below

**b** $\mathbf{R} = (4 + p)\mathbf{i} + (-5 + q)\mathbf{j}$, $4 + p = 3\lambda$ and $-5 + q = -\lambda$

$4 + p = 3(q - 5)$ so $p + 3q = 11$

**c** $2\sqrt{10} = 6.32$ newtons

3 **a** $\overrightarrow{PQ} = 9\mathbf{i} - 4\mathbf{j}$, $\overrightarrow{PR} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\overrightarrow{QR} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

**b** $|\overrightarrow{PQ}| = \sqrt{97}$, $|\overrightarrow{PR}| = \sqrt{59}$, $|\overrightarrow{QR}| = \sqrt{38}$ **c** 51.3°

4

184 (3 s.f.)

5 **a** (2, −7, −2) **b** rhombus **c** 36.1

$\overrightarrow{PQ} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$, $\overrightarrow{RS} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$, $\overrightarrow{TU} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

6 Let $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{TU}$ intersect at $X$: $\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$ for scalars $r$, $s$ and $t$

$\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $r = s = \frac{1}{2}$

$\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{b} + \mathbf{c}) + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{4}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $t = \frac{1}{2}$

So the line segments $PQ$, $RS$ and $TU$ meet at a point and bisect each other.

# Statistics & Mechanics 1. Sampling

| | | | |
|---|---|---|---|
| **1**<br><br>**Statistical sampling** | 1.1 | **Understand and use the terms 'population' and 'sample'.**<br><br>**Use samples to make informal inferences about the population.**<br><br>**Understand and use sampling techniques, including simple random sampling and opportunity sampling.**<br><br>**Select or critique sampling techniques in the context of solving a statistical problem, including understanding that different samples can lead to different conclusions about the population.** | **Students will be expected to comment on the advantages and disadvantages associated with a census and a sample.**<br><br>**Students will be expected to be familiar with: simple random sampling, stratified sampling, systematic sampling, quota sampling and opportunity (or convenience) sampling.** |

# Statistics & Mechanics 1. Sampling

**1** The table shows the daily mean temperature recorded on the first 15 days in May 1987 at Heathrow.

| **Day of month** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Daily mean temp (°C)** | 14.6 | 8.8 | 7.2 | 7.3 | 10.1 | 11.9 | 12.2 | 12.1 | 15.2 | 11.1 | 10.6 | 12.7 | 8.9 | 10.0 | 9.5 |



**a** Use an opportunity sample of the first 5 dates in the table to estimate the mean daily mean temperature at Heathrow for the first 15 days of May 1987.

**b** Describe how you could use the random number function on your calculator to select a simple random sample of 5 dates from this data.

**Hint** Make sure you describe your sampling frame.

**c** Use a simple random sample of 5 dates to estimate the mean daily mean temperature at Heathrow for the first 15 days of May 1987.

**d** Use all 15 dates to calculate the mean daily mean temperature at Heathrow for the first 15 days of May 1987. Comment on the reliability of your two samples.

**2 a** Give one advantage and one disadvantage of using:

**i** a census **ii** a sample survey.

**b** It is decided to take a sample of 100 from a population consisting of 500 elements. Explain how you would obtain a simple random sample from this population.

**3 a** Explain briefly what is meant by:

**i** a population **ii** a sampling frame.

**b** A market research organisation wants to take a sample of:

**i** owners of diesel motor cars in the UK

**ii** persons living in Oxford who suffered injuries to the back during July 1996.

Suggest a suitable sampling frame in each case.

**4** Write down one advantage and one disadvantage of using:

**a** stratified sampling **b** simple random sampling.

**5** The managing director of a factory wants to know what the workers think about the factory canteen facilities. 100 people work in the offices and 200 work on the shop floor.

The factory manager decides to ask the people who work in the offices.

**a** Suggest a reason why this is likely to produce a biased sample.

**b** Explain briefly how the factory manager could select a sample of 30 workers using:

**i** systematic sampling **ii** stratified sampling **iii** quota sampling.

**6** There are 64 girls and 56 boys in a school.

Explain briefly how you could take a random sample of 15 pupils using:

**a** simple random sampling **b** stratified sampling.

# Statistics & Mechanics 1. Sampling

## Answers

**1** **a** 9.6°C

**b** Sampling frame: first 15 days in May 1987
Allocate each date a number from 1 to 15
Use the random number function on calculator to generate 5 numbers between 1 and 15

**c** Students' own answers.

**d** 10.8°C

**2** **a** **i** Advantage: very accurate; disadvantage: expensive (time consuming).

**ii** Advantage: easier data collection (quick, cheap); disadvantage: possible bias.

**b** Assign unique 3-digit identifiers 000, 001, ..., 499 to each member of the population. Work along rows of random number tables generating 3-digit numbers. If these correspond to an identifier then include the corresponding member in the sample; ignore repeats and numbers greater than 499. Repeat this process until the sample contains 100 members.

**3** **a** **i** Collection of individual items.

**ii** List of sampling units.

**b** **i** List of registered owners from DVLA.

**ii** List of people visiting a doctor's clinic in Oxford in July 1996.

**4** **a** Advantage – the results are the most representative of the population since the structure of the sample reflects the structure of the population.
Disadvantage – you need to know the structure of the population before you can take a stratified sample.

**b** Advantage – quick and cheap.
Disadvantage – can introduce bias (e.g. if the sample, by chance, only includes very tall people in an investigation into heights of students).

**5** **a** People not in office not represented.

**b** **i** Get a list of the 300 workers at the factory.
$\frac{300}{30} = 10$ so choose one of the first ten workers on the list at random and every subsequent 10th worker on the list, e.g. if person 7 is chosen, then the sample includes workers 7, 17, 27, ..., 297.

**ii** The population contains 100 office workers ($\frac{1}{3}$ of population) and 200 shop floor workers ($\frac{2}{3}$ of population).
The sample should contain $\frac{1}{3} \times 30 = 10$ office workers and $\frac{2}{3} \times 30 = 20$ shop floor workers.
The 10 office workers in the sample should be a simple random sample of the 100 office workers.
The 20 shop floor workers should be a simple random sample of the 200 shop floor workers.

**iii** Decide the categories e.g. age, gender, office/non office and set a quota for each in proportion to their numbers in the population. Interview workers until quotas are full.

**6** **a** Allocate a number between 1 and 120 to each pupil.
Use random number tables, computer or calculator to select 15 different numbers between 1 and 120 (or equivalent).
Pupils corresponding to these numbers become the sample.

**b** Allocate numbers 1–64 to girls and 65–120 to boys.
Select $\frac{64}{120} \times 15 = 8$ different random numbers between 1 and 64 for girls.
Select 7 different random numbers between 65 and 120 for boys. Include the corresponding boys and girls in the sample.

# Statistics & Mechanics
## 2. Data Presentation and Interpretation

| | | | |
|---|---|---|---|
| **2**<br><br>**Data presentation and interpretation** | 2.1 | **Interpret diagrams for single-variable data, including understanding that area in a histogram represents frequency.**<br><br>**Connect to probability distributions.** | **Students should be familiar with histograms, frequency polygons, box and whisker plots (including outliers) and cumulative frequency diagrams.** |

| | | | |
|---|---|---|---|
| **2**<br><br>**Data presentation and interpretation**<br><br>*continued* | 2.2 | **Interpret scatter diagrams and regression lines for bivariate data, including recognition of scatter diagrams which include distinct sections of the population (calculations involving regression lines are excluded).** | **Students should be familiar with the terms explanatory (independent) and response (dependent) variables.**<br><br>Use to make predictions within the range of values of the explanatory variable and the dangers of extrapolation. Derivations will not be required. Variables other than $x$ and $y$ may be used.<br><br>**Use of interpolation and the dangers of extrapolation. Variables other than $x$ and $y$ may be used.**<br><br>Change of variable may be required, e.g. using knowledge of logarithms to reduce a relationship of the form $y = ax^n$ or $y = kb^x$ into linear form to estimate $a$ and $n$ or $k$ and $b$. |
| | | **Understand informal interpretation of correlation.**<br><br>**Understand that correlation does not imply causation.** | **Use of terms such as positive, negative, zero, strong and weak are expected.** |

| | | | |
|---|---|---|---|
| | 2.3 | Interpret measures of central tendency and variation, extending to standard deviation. | Data may be discrete, continuous, grouped or ungrouped. Understanding and use of coding.<br><br>Measures of central tendency: mean, median, mode.<br><br>Measures of variation: variance, standard deviation, range and interpercentile ranges.<br><br>Use of linear interpolation to calculate percentiles from grouped data is expected. |
| | | Be able to calculate standard deviation, including from summary statistics. | Students should be able to use the statistic<br><br>$S_{xx} = \sum(x-\bar{x})^2 = \sum x^2 - \frac{(\sum x)^2}{n}$<br><br>Use of standard deviation $= \sqrt{\frac{S_{xx}}{n}}$ (or equivalent) is expected but the use of $S = \sqrt{\frac{S_{xx}}{n-1}}$ (as used on spreadsheets) will be accepted. |

| | | | |
|---|---|---|---|
| **2**<br><br>**Data presentation and interpretation**<br><br>*continued* | 2.4 | Recognise and interpret possible outliers in data sets and statistical diagrams. | Any rule needed to identify outliers will be specified in the question.<br><br>For example, use of $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$ or mean $\pm 3 \times$ standard deviation. |
| | | Select or critique data presentation techniques in the context of a statistical problem. | Students will be expected to draw simple inferences and give interpretations to measures of central tendency and variation. Significance tests, other than those mentioned in Section 5, will not be expected. |
| | | Be able to clean data, including dealing with missing data, errors and outliers. | For example, students may be asked to identify possible outliers on a box plot or scatter diagram. |

# Statistics & Mechanics
# 2. Data Presentation and Interpretation

E/P 1 The table gives the distances travelled to school, in km, of the population of children in a particular region of the United Kingdom.

| Distance, $d$ (km) | $0 \leqslant d < 1$ | $1 \leqslant d < 2$ | $2 \leqslant d < 3$ | $3 \leqslant d < 5$ | $5 \leqslant d < 10$ | $10 \leqslant d$ |
|---|---|---|---|---|---|---|
| Number | 2565 | 1784 | 1170 | 756 | 630 | 135 |

A histogram of this data was drawn with distance along the horizontal axis. A bar of horizontal width 1.5 cm and height 5.7 cm represented the 0–1 km group.

Find the widths and heights, in cm, to one decimal place, of the bars representing the following groups:

**a** $2 \leqslant d < 3$ **b** $5 \leqslant d < 10$ **(5 marks)**

E 2 A manufacturer stores drums of chemicals. During storage, evaporation take place. A random sample of 10 drums was taken and the time in storage, $x$ weeks, and the evaporation loss, $y$ ml, are shown in the table below.

| $x$ | 3 | 5 | 6 | 8 | 10 | 12 | 13 | 15 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 36 | 50 | 53 | 61 | 69 | 79 | 82 | 90 | 88 | 96 |

**a** On graph paper, draw a scatter diagram to represent these data. **(3)**

**b** Give a reason to support fitting a regression model of the form $y = a + bx$ to these data. **(1)**

The equation of the regression line of $y$ on $x$ is $y = 29.02 + 3.9x$.

**c** Give an interpretation of the value of the gradient in the equation of the regression line. **(1)**

The manufacturer uses this model to predict the amount of evaporation that would take place after 19 weeks and after 35 weeks.

**d** Comment, with a reason, on the reliability of each of these predictions. **(2)**

E 3 20 endangered forest owlets were caught for ringing. Their wingspans ($x$ cm) were measured to the nearest centimetre.

The following summary statistics were worked out:

$\Sigma x = 316 \qquad \Sigma x^2 = 5078$

**a** Work out the mean and the standard deviation of the wingspans of the 20 birds. **(3 marks)**

One more bird was caught. It had a wingspan of 13 centimetres.

**b** Without doing any further calculation, say how you think this extra wingspan will affect the mean wingspan. **(1 mark)**

20 giant ibises were also caught for ringing. Their wingspans ($y$ cm) were also measured to the nearest centimetre and the data coded using $z = \frac{y-5}{10}$.

The following summary statistics were obtained from the coded data:

$\Sigma z = 104 \qquad S_{zz} = 1.8$

**c** Work out the mean and standard deviation of the wingspans of the giant ibis. **(5 marks)**

4 A frequency distribution is shown below.

| Class interval | 1–10 | 11–20 | 21–30 | 31–40 | 41–50 |
|---|---|---|---|---|---|
| Frequency | 10 | 20 | 30 | 24 | 16 |

**a** Use interpolation to estimate the value of the 30th percentile.

**b** Use interpolation to estimate the value of the 70th percentile.

**c** Hence estimate the 30% to 70% interpercentile range.

E 5 The table shows some data collected on the temperature in °C of a chemical reaction ($t$) and the amount of dry residue produced ($d$ grams).

| Temperature, $t$ (°C) | 38 | 51 | 72 | 83 | 89 | 94 |
|---|---|---|---|---|---|---|
| Dry residue, $d$ (grams) | 4.3 | 11.7 | 58.6 | 136.7 | 217.0 | 318.8 |

The data are coded using the changes of variable $x = t$ and $y = \log d$. The regression line of $y$ on $x$ is found to be $y = -0.635 + 0.0334x$.

**a** Given that the data can be modelled by an equation of the form $d = ab^t$ where $a$ and $b$ are constants, find the values of $a$ and $b$. **(3 marks)**

**b** Explain why this model is not reliable for estimating the amount of dry residue produced when the temperature is 151 °C. **(1 mark)**

E 6 Energy consumption is claimed to be a good predictor of Gross National Product.
An economist recorded the energy consumption ($x$) and the Gross National Product ($y$) for eight countries. The data is shown in the table.

| **Energy consumption ($x$)** | 3.4 | 7.7 | 12.0 | 75 | 58 | 67 | 113 | 131 |
|---|---|---|---|---|---|---|---|---|
| **Gross National Product ($y$)** | 55 | 240 | 390 | 1100 | 1390 | 1330 | 1400 | 1900 |

The equation of the regression line of $y$ on $x$ is $y = 225 + 12.9x$.

The economist uses this regression equation to estimate the energy consumption of a country with a Gross National Product of 3500.

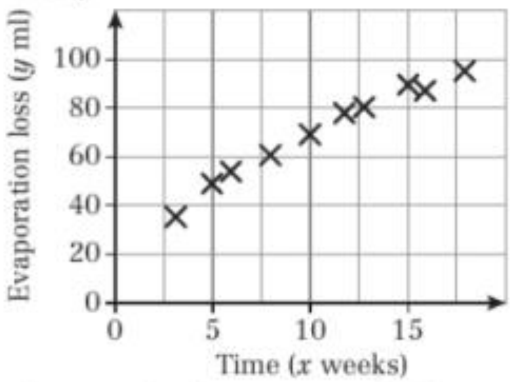
Give two reasons why this may not be a valid estimate. **(2 marks)**

## Statistics & Mechanics
## 2. Data Presentation and Interpretation

## Answers

1 **a** width = 1.5 cm, height = 2.6 cm
**b** width = 7.5 cm, height = 0.28 cm

2 **a**




**b** The points lie close to a straight line.
**c** 3.90 ml of the chemicals evaporate each week.
**d** The estimate for 19 weeks is reasonably reliable, since it is just outside the range of the data. The estimate for 35 weeks is unreliable, since it is far outside the range of the data.

3 **a** Mean 15.8 cm, standard deviation 2.06 cm
**b** The mean wingspan will decrease.
**c** Mean 57 cm, standard deviation 3 cm

4 **a** 20.5 **b** 34.7 **c** 14.2

5 **a** $a = 0.232$ (3 s.f.), $b = 1.08$ (3 s.f.)
**b** 151 °C is outside the range of the data (extrapolation).

6 (1) 3500 is outside the range of the data (extrapolation).
(2) The regression equation should only be used to predict a value of GNP ($y$) given energy consumption ($x$).

# Statistics & Mechanics 3. Probability

| | | | |
|---|---|---|---|
| **3**<br><br>**Probability** | 3.1 | **Understand and use mutually exclusive and independent events when calculating probabilities.** | **Venn diagrams or tree diagrams may be used.** Set notation to describe events may be used.<br><br>Use of $\mathrm{P}(B \mid A) = \mathrm{P}(B)$, $\mathrm{P}(A \mid B) = \mathrm{P}(A)$,<br><br>$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$ in connection with independent events. |
| | | **Link to discrete and continuous distributions.** | **No formal knowledge of probability density functions is required but students should understand that area under the curve represents probability in the case of a continuous distribution.** |
| | 3.2 | Understand and use conditional probability, including the use of tree diagrams, Venn diagrams, two-way tables.<br><br>Understand and use the conditional probability formula<br><br>$\mathrm{P}(A\|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ | Understanding and use of<br><br>$\mathrm{P}(A') = 1 - \mathrm{P}(A)$,<br><br>$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$,<br><br>$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B \mid A)$. |

| | | | |
|---|---|---|---|
| **3**<br><br>**Probability**<br><br>*continued* | 3.3 | Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions. | For example, questioning the assumption that a die or coin is fair. |

# Statistics & Mechanics 3. Probability

(P) 1 The scores of 250 students in a test are recorded in a table.

| Score, $s$ | Frequency (male) | Frequency (female) |
|---|---|---|
| $20 \leqslant s < 25$ | 7 | 8 |
| $25 \leqslant s < 30$ | 15 | 13 |
| $30 \leqslant s < 35$ | 18 | 19 |
| $35 \leqslant s < 40$ | 25 | 30 |
| $40 \leqslant s < 45$ | 30 | 26 |
| $45 \leqslant s < 50$ | 27 | 32 |

One student is chosen at random.

**a** Find the probability that the student is female.

**b** Find the probability that the student scored less than 35.

**c** Find the probability that the student is male with a score $s$ such that $25 \leq s < 35$.

In order to pass the test, students must score 37 or more.

**d** Estimate the probability that a student chosen at random passes the test. State one assumption you have made in making your estimate.

(E/P) 2 For events $J$ and $K$, P($J$ or $K$ or both) = 0.5, P($K$ but not $J$) = 0.2 and P($J$ but not $K$) = 0.25.

**a** Draw a Venn diagram to represent events $J$ and $K$ and the sample space $S$. **(3 marks)**

**b** Determine whether events $J$ and $K$ are independent. **(3 marks)**

(P) 3 In a factory, machines $A$, $B$ and $C$ produce electronic components. Machine $A$ produces 16% of the components, machine $B$ produces 50% of the components and machine $C$ produces the rest. Some of the components are defective. Machine $A$ produces 4%, machine $B$ 3% and machine $C$ 7% defective components.

**a** Draw a tree diagram to represent this information.

**b** Find the probability that a randomly selected component is:

**i** produced by machine $B$ and is defective **ii** defective.

(E/P) 4 $J$, $K$ and $L$ are three events such that $P(J) = 0.25$, $P(K) = 0.45$ and $P(L) = 0.15$. Given that $K$ and $L$ are independent, $J$ and $L$ are mutually exclusive and $P(J \cap K) = 0.1$

**a** draw a Venn diagram to illustrate this situation. **(2 marks)**

**b** Find:

**i** $P(J \cup K)$ **(1 mark)**

**ii** $P(J' \cap L')$ **(1 mark)**

**iii** $P(J|K)$ **(2 marks)**

**iv** $P(K|J' \cap L')$ **(2 marks)**

(E) 5 On a randomly chosen day the probabilities that Bill travels to work by car, by bus or by train are 0.1, 0.6 and 0.3 respectively. The probabilities of being late when using these methods of travel are 0.55, 0.3 and 0.05 respectively.

**a** Draw a tree diagram to represent this information. **(3 marks)**

**b** Find the probability that on a randomly chosen day,

**i** Bill travels by train and is late **(2 marks)**

**ii** Bill is late. **(2 marks)**

**c** Given that Bill is late, find the probability that he did not travel by car. **(4 marks)**

(E/P) 6 A box $A$ contains 7 counters of which 4 are green and 3 are blue.
A box $B$ contains 5 counters of which 2 are green and 3 are blue.
A counter is drawn at random from box $A$ and placed in box $B$. A second counter is drawn at random from box $A$ and placed in box $B$.
A third counter is then drawn at random from the counters in box $B$.

**a** Draw a tree diagram to show this situation. **(4 marks)**

The event $C$ occurs when the 2 counters drawn from box $A$ are of the same colour.
The event $D$ occurs when the counter drawn from box $B$ is blue.

**b** Find $P(C)$. **(3 marks)**

**c** Show that $P(D) = \frac{27}{49}$ **(3 marks)**

**d** Show that $P(C \cap D) = \frac{11}{49}$ **(2 marks)**

**e** Hence find $P(C \cup D)$. **(2 marks)**

**f** Given that all three counters drawn are the same colour, find the probability that they are all green. **(3 marks)**

# Statistics & Mechanics 3. Probability

## Answers

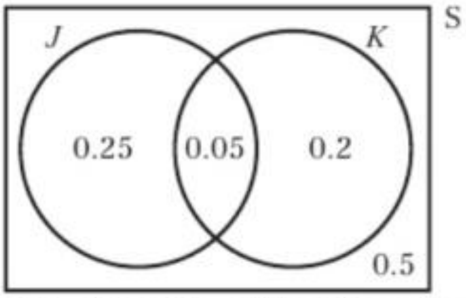
1 **a** $\frac{64}{125}$

**b** $\frac{8}{25}$

**c** $\frac{33}{250}$

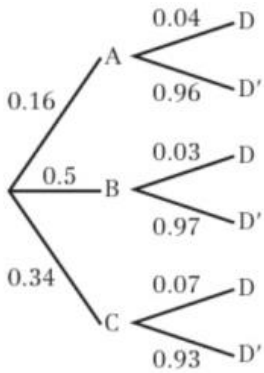
**d** $\frac{74}{125}$, using interpolation and assuming uniform distribution of scores

2 **a**



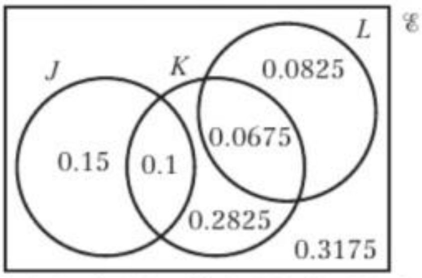

**b** $P(J) = 0.3$, $P(K) = 0.25$, $P(J \text{ and } K) = 0.05$
$P(J) \times P(K) = 0.075 \neq P(J \text{ and } K)$, so $J$ and $K$ are not independent.

3 **a**




**b** **i** 0.015 **ii** 0.0452

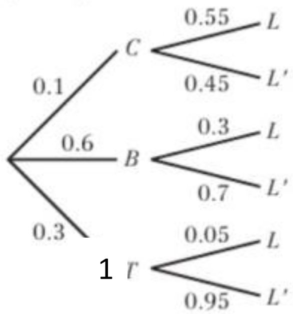
4 **a**




**b** **i** 0.6 **ii** 0.6 **iii** 0.222 (3 s.f.) **iv** 0.471 (3 s.f.)

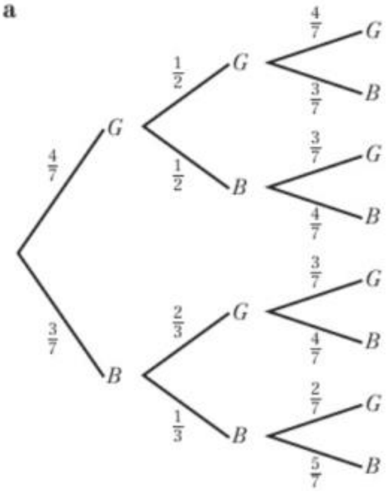
# Statistics & Mechanics 3. Probability

5 **a**




**b** **i** 0.015 **ii** 0.25 **c** 0.78

6 **a**




**b** $\frac{3}{7}$

**c** Adding together the probabilities on the 4 branches of the tree diagram where the counter from box $B$ is blue: $\frac{12}{98} + \frac{16}{98} + \frac{24}{147} + \frac{15}{147} = \frac{27}{49}$

**d** Adding together the probabilities on the 2 branches of the tree diagram where events $C$ and $D$ both occur. $\frac{12}{98} + \frac{15}{147} = \frac{11}{49}$

**e** $\frac{37}{49}$ **f** $\frac{8}{13}$

# Statistics & Mechanics 4. Distributions

| | | | |
|---|---|---|---|
| **4**<br><br>**Statistical distributions** | 4.1 | **Understand and use simple, discrete probability distributions (calculation of mean and variance of discrete random variables is excluded), including the binomial distribution, as a model; calculate probabilities using the binomial distribution.** | **Students will be expected to use distributions to model a real-world situation and to comment critically on the appropriateness.**<br><br>**Students should know and be able to identify the discrete uniform distribution.**<br><br>**The notation $X \sim \mathrm{B}(n, p)$ may be used.**<br><br>**Use of a calculator to find individual or cumulative binomial probabilities.** |
| | 4.2 | Understand and use the Normal distribution as a model; find probabilities using the Normal distribution | The notation $X \sim \mathrm{N}(\mu, \sigma^2)$ may be used.<br><br>Knowledge of the shape and the symmetry of the distribution is required. Knowledge of the probability density function is not required. Derivation of the mean, variance and cumulative distribution function is not required.<br><br>Questions may involve the solution of simultaneous equations.<br><br>Students will be expected to use their calculator to find probabilities connected with the normal distribution. |
| | | Link to histograms, mean, standard deviation, points of inflection | Students should know that the points of inflection on the normal curve are at $x = \mu \pm \sigma$.<br><br>The derivation of this result is not expected. |
| | | and the binomial distribution. | Students should know that when $n$ is large and $p$ is close to 0.5 the distribution $\mathrm{B}(n, p)$ can be approximated by $\mathrm{N}(np, np[1 - p])$<br><br>The application of a continuity correction is expected. |

| | | | |
|---|---|---|---|
| **4**<br><br>**Statistical distributions**<br><br>*continued* | 4.3 | Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the binomial or Normal model may not be appropriate. | Students should know under what conditions a binomial distribution or a Normal distribution might be a suitable model. |

# Statistics & Mechanics 4. Distributions

E/P 1 The random variable $X$ has probability function

$$P(X = x) = \frac{(3x - 1)}{26} \qquad x = 1, 2, 3, 4.$$

**a** Construct a table giving the probability distribution of $X$. **(2 marks)**

**b** Find $P(2 < X \leqslant 4)$. **(2 marks)**

P 2 Records kept in a hospital show that 3 out of every 10 patients who visit the accident and emergency department have to wait more than half an hour. Find, to 3 decimal places, the probability that of the first 12 patients who come to the accident and emergency department:

**a** none

**b** more than 2

will have to wait more than half an hour.

P 3 A completely unprepared student is given a true/false-type test with 10 questions. Assuming that the student answers all the questions at random:

**a** find the probability that the student gets all the answers correct.

It is decided that a pass will be awarded for 8 or more correct answers.

**b** Find the probability that the student passes the test.

E/P 4 The time a mobile phone battery lasts before needing to be recharged is assumed to be normally distributed with a mean of 48 hours and a standard deviation of 8 hours.

**a** Find the probability that a battery will last for more than 60 hours. **(2 marks)**

**b** Find the probability that the battery lasts less than 35 hours. **(1 mark)**

A random sample of 30 phone batteries is taken.

**c** Find the probability that 3 or fewer last less than 35 hours. **(2 marks)**

E 5 The owner of a local corner shop calculates that the probability of a customer buying a newspaper is 0.40.

A random sample of 100 customers is recorded.

**a** Give two reasons why a normal approximation may be used in this situation. **(2 marks)**

**b** Write down the parameters of the normal distribution used. **(2 marks)**

**c** Use this approximation to estimate the probability that at least half the customers bought a newspaper. **(2 marks)**

E/P 6 A herbalist claims that a particular remedy is successful in curing a particular disease in 52% of cases.

A random sample of 25 people who took the remedy is taken.

**a** Find the probability that more than 12 people in the sample were cured. **(2 marks)**

A second random sample of 300 people was taken and 170 were cured.

**b** Assuming the herbalist's claim is true, use a suitable approximation to find the probability that at least 170 people were cured. **(4 marks)**

**c** Using your answer to part **b**, comment on the herbalist's claim. **(1 mark)**

# Statistics & Mechanics 4. Distributions

## Answers

1 **a**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | 0.0769 | 0.1923 | 0.3077 | 0.4231 |

**b** $\frac{19}{26}$

2 **a** 0.014 (3 d.p.) **b** 0.747 (3 d.p.)

3 **a** 0.000977 **b** 0.0547

4 **a** 0.0668 **b** 0.0521 **c** 0.9315

5 **a** $n$ is large and $p$ is close to 0.5.
**b** $\mu = 40$, $\sigma^2 = 24$
**c** 0.0262

6 **a** 0.5801 **b** 0.0594
**c** Assuming that the claim is correct, there is a greater than 5% chance that 170 people out of 300 would be cured, therefore there is insufficient evidence to reject the herbalist's claim.

# Statistics & Mechanics 5. Hypothesis Testing

| | | | |
|---|---|---|---|
| **5**<br><br>**Statistical hypothesis testing** | 5.1 | **Understand and apply the language of statistical hypothesis testing, developed through a binomial model: null hypothesis, alternative hypothesis, significance level, test statistic, 1-tail test, 2-tail test, critical value, critical region, acceptance region, *p*-value;**<br><br>extend to correlation coefficients as measures of how close data points lie to a straight line.<br><br>and<br><br>be able to interpret a given correlation coefficient using a given *p*-value or critical value (calculation of correlation coefficients is excluded). | **An informal appreciation that the expected value of a binomial distribution is given by $np$ may be required for a 2-tail test.**<br><br>Students should know that the product moment correlation coefficient $r$ satisfies $\|r\| \le 1$ and that a value of $r = \pm 1$ means the data points all lie on a straight line.<br><br>Students will be expected to calculate a value of $r$ using their calculator but use of the formula is not required.<br><br>Hypotheses should be stated in terms of $\rho$ with a null hypothesis of $\rho = 0$ where $\rho$ represents the population correlation coefficient.<br><br>Tables of critical values or a *p*-value will be given. |
| **5**<br><br>**Statistical hypothesis testing**<br><br>*continued* | 5.2 | **Conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context.** | |
| | | **Understand that a sample is being used to make an inference about the population.**<br><br>**and**<br><br>**appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.** | **Hypotheses should be expressed in terms of the population parameter *p***<br><br>**A formal understanding of Type I errors is not expected.** |
| | 5.3 | Conduct a statistical hypothesis test for the mean of a Normal distribution with known, given or assumed variance and interpret the results in context. | Students should know that:<br><br>If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ and<br><br>that a test for $\mu$ can be carried out using: $\frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0, 1^2)$.<br><br>No proofs required.<br><br>Hypotheses should be stated in terms of the population mean $\mu$.<br><br>Knowledge of the Central Limit Theorem or other large sample approximations is not required. |

P

# Statistics & Mechanics 5. Hypothesis Testing

E/P 1 The manager of a superstore thinks that the probability of a person buying a certain make of computer is only 0.2.
To test whether this hypothesis is true the manager decides to record the make of computer bought by a random sample of 50 people who bought a computer.

**a** Find the critical region that would enable the manager to test whether or not there is evidence that the probability is different from 0.2. The probability of each tail should be as close to 2.5% as possible. **(4 marks)**

**b** Write down the significance level of this test. **(2 marks)**

15 people buy that certain make.

**c** Comment on this observation in light of your critical region. **(2 marks)**

E/P 2 A pharmaceutical company claims that 85% of patients suffering from a chronic rash recover when treated with a new ointment.

A random sample of 20 patients with this rash is taken from hospital records.

**a** Write down a suitable distribution to model the number of patients in this sample who recover when treated with the new ointment. **(2 marks)**

**b** Given that the claim is correct, find the probability that the ointment will be successful for exactly 16 patients. **(2 marks)**

The hospital believes that the claim is incorrect and the percentage who will recover is lower. From the records an administrator took a random sample of 30 patients who had been prescribed the ointment. She found that 20 had recovered.

**c** Stating your hypotheses clearly, test, at the 5% level of significance, the hospital's belief. **(6 marks)**

E 3 As part of a survey in a particular profession, age, $x$ years, and yearly salary, £$y$ thousands, were recorded. The values of $x$ and $y$ for a randomly selected sample of ten members of the profession are as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 30 | 52 | 38 | 48 | 56 | 44 | 41 | 25 | 32 | 27 |
| $y$ | 22 | 38 | 40 | 34 | 35 | 32 | 28 | 27 | 29 | 41 |

**a** Calculate, to 3 decimal places, the product moment correlation coefficient between age and salary. **(1 mark)**

It is suggested that there is no correlation between age and salary.

**b** Test this suggestion at the 5% significance level, stating your null and alternative hypotheses clearly. **(3 marks)**

E 4 A meteorologist believes that there is a positive correlation between daily mean windspeed and daily maximum gust. She collects data from the large data set for 5 days during August 2015 in the town of Hurn.

| | | | | | |
|---|---|---|---|---|---|
| **Mean windspeed (knots)** | 4 | 7 | 7 | 8 | 5 |
| **Daily maximum gust (knots)** | 14 | 22 | 18 | 20 | 17 |



By calculating the product moment correlation coefficient for these data, test at the 5% level of significance whether there is evidence to support the meteorologist's claim. State your hypotheses clearly. **(4 marks)**

E 5 Climbing rope produced by a manufacturer is known to be such that one-metre lengths have breaking strengths that are normally distributed with mean 170.2 kg and standard deviation 10.5 kg. Find, to 3 decimal places, the probability that:

**a** a one-metre length of rope chosen at random from those produced by the manufacturer will have a breaking strength of 175 kg to the nearest kg **(2 marks)**

**b** a random sample of 50 one-metre lengths will have a mean breaking strength of more than 172.4 kg. **(3 marks)**

A new component material is added to the ropes being produced. The manufacturer believes that this will increase the mean breaking strength without changing the standard deviation. A random sample of 50 one-metre lengths of the new rope is found to have a mean breaking strength of 172.4 kg.

**c** Perform a significance test at the 5% level to decide whether this result provides sufficient evidence to confirm the manufacturer's belief that the mean breaking strength is increased. State clearly the null and alternative hypotheses that you are using. **(3 marks)**

E/P 6 The random variable $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$.

**a** Write down the distribution of the sample mean $\overline{X}$ of a random sample of size $n$. **(1 mark)**

A construction company wishes to determine the mean time taken to drill a fixed number of holes in a metal sheet.

**b** Determine how large a random sample is needed so that the expert can be 95% certain that the sample mean time will differ from the true mean time by less than 15 seconds. Assume that it is known from previous studies that $\sigma = 40$ seconds. **(4 marks)**

# Statistics & Mechanics 5. Hypothesis Testing

## Answers

1 **a** Critical region is $X \leqslant 4$ and $X \geqslant 16$
**b** 0.0493
**c** There is insufficient evidence to reject $H_0$.
There is no evidence to suggest that the proportion of people buying that certain make of computer differs from 0.2.

2 **a** $X \sim B(20, 0.85)$
**b** 0.1821
**c** Test statistic is proportion of patients who recover.
$H_0$: $p = 0.85$, $H_1$: $p < 0.85$
$P(X \leqslant 20) = 0.00966$
$0.00966 < 0.05$ so there is enough evidence to reject $H_0$. The percentage of patients who recover after treatment with the new ointment is lower than 85%.

3 **a** 0.340 (3 d.p.)
**b** $H_0$: $\rho = 0$, $H_1$: $\rho \neq 0$, critical value = $\pm 0.6319$. Accept $H_0$. There is not enough evidence that there is a correlation between age and salary.

4 $r = 0.843$ (3 s.f.), $H_0$: $\rho = 0$, $H_1$: $\rho > 0$, critical value 0.8054. Reject $H_0$. There is evidence that mean windspeed and daily maximum gust are positively correlated.

5 **a** 0.0342 **b** Accept $\leqslant 0.069$
**c** Test statistic $= 1.4815\ldots < 1.6449$
Not significant so accept $H_0$. Insufficient evidence of an increase in the mean breaking strength of climbing rope.

6 **a** $X \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ **b** Need $n = 28$ or more

# Statistics & Mechanics 6. Units (Modelling)

| | | | |
|---|---|---|---|
| **6**<br><br>**Quantities and units in mechanics** | 6.1 | **Understand and use fundamental quantities and units in the S.I. system: length, time, mass.**<br><br>**Understand and use derived quantities and units: velocity, acceleration, force, weight, moment.** | Students may be required to convert one unit into another e.g. km $h^{-1}$ into m $s^{-1}$ |

Ⓟ 1 A diver dives from a diving board into a swimming pool with a depth of 4.5 m. The height of the diver above the water, $h$ m, can be modelled using $h = 10 - 0.58x^2$ for $0 \leqslant x \leqslant 5$, where $x$ m is the horizontal distance from the end of the diving board.

**a** Find the height of the diver when $x = 2$ m.

**b** Find the horizontal distance from the end of the diving board to the point where the diver enters the water.

In this model the diver is modelled as particle.

**c** Describe the effects of this modelling assumption.

**d** Comment on the validity of this modelling assumption for the motion of the diver after she enters the water.

2 A man throws a bowling ball in a bowling alley.

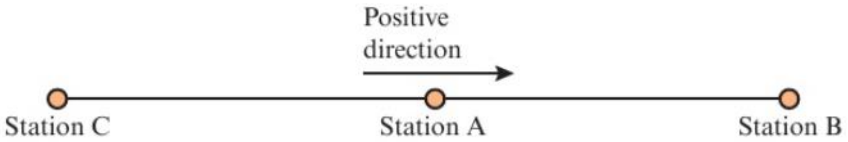
**a** Make a list of the assumptions you might make to create a simple model of the motion of the bowling ball.

**b** Taking the direction that the ball travels in as the positive direction, state with a reason whether each of the following are likely to be positive or negative:

**i** the velocity **ii** the acceleration.

# Statistics & Mechanics 6. Units (Modelling)

3 A train engine pulling a truck starts at station $A$ then travels in a straight line to station $B$. It then moves back from station $B$ to station $A$ and on to station $C$ as shown in the diagram.




**Hint** The **sign** of something means whether it is positive or negative.

What is the sign of the velocity and displacement on the journey from:

**a** station $A$ to station $B$ **b** station $B$ to station $A$ **c** station $A$ to station $C$?

4 The acceleration of a boat is given by $\mathbf{a} = -0.05\mathbf{i} + 0.15\mathbf{j}\ \text{m s}^{-2}$. Find:

**a** the magnitude of the acceleration

**b** the angle the direction of the acceleration vector makes with the unit vector $\mathbf{i}$.

5 The velocity of a toy car is given by $\mathbf{v} = 3.5\mathbf{i} - 2.5\mathbf{j}\ \text{m s}^{-1}$. Find:

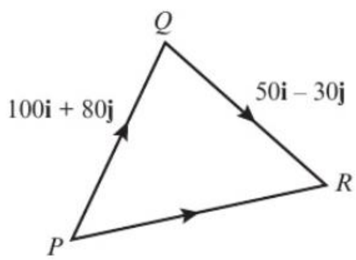
**a** the speed of the toy car

**b** the angle the direction of motion of the toy car makes with the unit vector $\mathbf{j}$.

6 A plane flies from $P$ to $Q$ and then from $Q$ to $R$.

The displacement from $P$ to $Q$ is $100\mathbf{i} + 80\mathbf{j}$ m.

The displacement from $Q$ to $R$ is $50\mathbf{i} - 30\mathbf{j}$ m.




**a** Find the magnitude of the displacement from $P$ to $R$.

**b** Find the total distance the plane has travelled in getting from $P$ to $R$.

**c** Find the angle the vector $\overrightarrow{PQ}$ makes with the unit vector $\mathbf{j}$.

# Statistics & Mechanics 6. Units (Modelling)

## Answers

1 **a** 7.68 m **b** 4.15 m
**c** Ignore the effects of air resistance on the diver and rotational effects of external forces.
**d** Assumption not valid, diver experiences drag and buoyancy in the water.

2 **a** Model ball as a particle. Assume the floor is smooth.
**b** **i** Positive – the positive direction is defined as the direction in which the ball is travelling.
**ii** Negative – the ball will be slowing down.

3 **a** Velocity is positive, displacement is positive
**b** Velocity is negative, displacement is positive
**c** Velocity is negative, displacement is negative

4 **a** $0.158\,\text{ms}^{-2}$ **b** 108.4°

5 **a** $4.3\,\text{ms}^{-1}$ **b** 125.5°

6 **a** 158.1 m **b** 186.4 m **c** 51.3°

# Statistics & Mechanics 7. Kinematics

| | | | |
|---|---|---|---|
| **7**<br>**Kinematics** | 7.1 | **Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.** | Students should know that distance and speed must be positive. |
| | 7.2 | **Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph.** | **Graphical solutions to problems may be required.** |
| | 7.3 | **Understand, use and derive the formulae for constant acceleration for motion in a straight line.**<br><br>Extend to 2 dimensions using vectors. | **Derivation may use knowledge of sections 7.2 and/or 7.4**<br><br>**Understand and use *suvat* formulae for constant acceleration in 2-D,**<br><br>e.g. $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ , $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ with vectors given in $\mathbf{i} - \mathbf{j}$ or column vector form.<br><br>Use vectors to solve problems. |
| | 7.4 | **Use calculus in kinematics for motion in a straight line:**<br><br>$v = \frac{\mathrm{d}r}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2}$<br><br>$r = \int v \; dt$ , $v = \int a \; dt$<br><br>Extend to 2 dimensions using vectors. | **The level of calculus required will be consistent with that in Sections 7 and 8 in Paper 1** and Sections 6 and 7 in Paper 2.<br><br>Differentiation and integration of a vector with respect to time. e.g.<br><br>Given $\mathbf{r} = t^2\mathbf{i} + t^{\frac{3}{2}}\mathbf{j}$, find $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ at a given time. |
| | 7.5 | Model motion under gravity in a vertical plane using vectors; projectiles. | Derivation of formulae for time of flight, range and greatest height and the derivation of the equation of the path of a projectile may be required. |

# Statistics & Mechanics 7. Kinematics

E/P 1 A train starts from rest at station $A$ and accelerates uniformly at $3x\,\text{m s}^{-2}$ until it reaches a velocity of $30\,\text{m s}^{-1}$. For the next $T$ seconds the train maintains this constant velocity. The train then decelerates uniformly at $x\,\text{m s}^{-2}$ until it comes to rest at a station $B$. The distance between the stations is 6 km and the time taken from $A$ to $B$ is 5 minutes.

**a** Sketch a velocity–time graph to illustrate this journey. **(2 marks)**

**b** Show that $\frac{40}{x} + T = 300$. **(4 marks)**

**c** Find the value of $T$ and the value of $x$. **(2 marks)**

**d** Calculate the distance the train travels at constant velocity. **(2 marks)**

**e** Calculate the time taken from leaving $A$ until reaching the point halfway between the stations. **(3 marks)**

P 2 At a time $t$ seconds after launch, the space shuttle can be modelled as a particle moving in a straight line with acceleration, $a\,\text{m s}^{-2}$, given by the equation:

$$a = (6.77 \times 10^{-7})t^3 - (3.98 \times 10^{-4})t^2 + 0.105t + 0.859, \quad 124 \leqslant t \leqslant 446$$

**a** Suggest two reasons why the space shuttle might experience variable acceleration during its launch phase.

Given that the velocity of the space shuttle at time $t = 124$ is $974\,\text{m s}^{-1}$:

**b** find an expression for the velocity $v\,\text{m s}^{-1}$ of the space shuttle at time $t$. Give your coefficients to 3 significant figures.

**c** Hence find the velocity of the space shuttle at time $t = 446$, correct to 3 s.f.

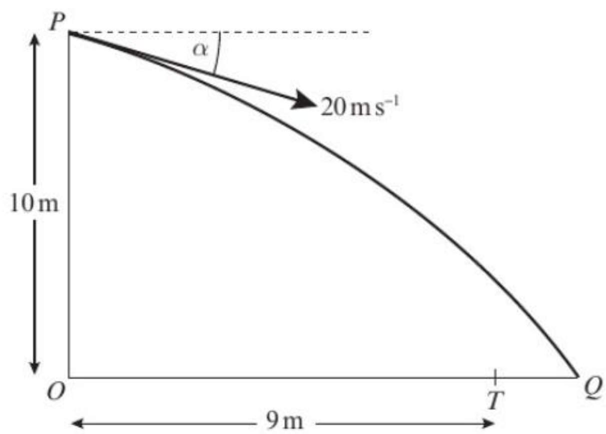
From $t = 446$, the space shuttle maintains a constant acceleration of $28.6\,\text{m s}^{-2}$ until it reaches its escape velocity of $7.85\,\text{km s}^{-1}$. It then cuts its main engines.

**d** Calculate the time at which the space shuttle cuts its main engines.

E/P 3 In this question use $g = 10\,\text{m s}^{-2}$.

A stone is thrown from a point $P$ at a target, which is on horizontal ground. The point $P$ is 10 m above the point $O$ on the ground. The stone is thrown from $P$ with speed $20\,\text{m s}^{-1}$ at an angle of $\alpha$ below the horizontal, where $\tan\alpha = \frac{3}{4}$.

The stone is modelled as a particle and the target as a point $T$. The distance $OT$ is 9 m. The stone misses the target and hits the ground at the point $Q$, where $OTQ$ is a straight line, as shown in the diagram. Find:




**a** the time taken by the ball to travel from $P$ to $Q$ **(5 marks)**

**b** the distance $TQ$. **(4 marks)**

The point $A$ is on the path of the ball vertically above $T$.

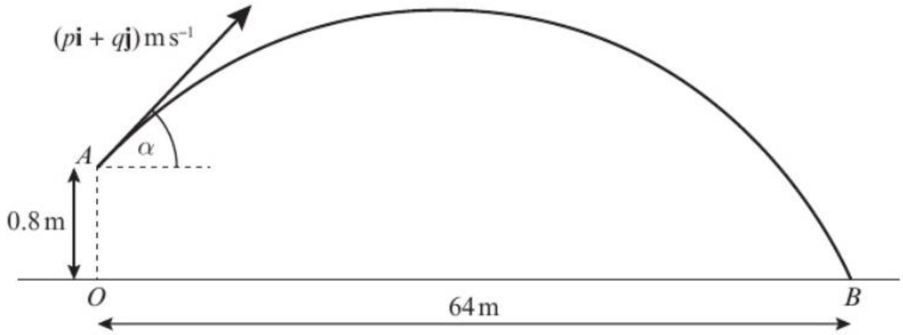
**c** Find the speed of the ball at $A$. **(5 marks)**

E/P 4 A vertical mast is 32 m high. Two balls $P$ and $Q$ are projected simultaneously. Ball $P$ is projected horizontally from the top of the mast with speed $18\,\text{m s}^{-1}$. Ball $Q$ is projected from the bottom of the mast with speed $30\,\text{m s}^{-1}$ at an angle $\alpha$ above the horizontal. The balls move freely under gravity in the same vertical plane and collide in mid-air. By considering the horizontal motion of each ball,

**a** prove that $\cos\alpha = \frac{3}{5}$ **(4 marks)**

**b** Find the time which elapses between the instant when the balls are projected and the instant when they collide. **(4 marks)**

E/P 5




A cricket ball is hit from a point $A$ with velocity of $(p\mathbf{i} + q\mathbf{j})\,\text{m s}^{-1}$, at an angle $\alpha$ above the horizontal. $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors horizontally and vertically upwards respectively. The point $A$ is 0.8 m vertically above the point $O$, which is on horizontal ground.

The ball takes 4 seconds to travel from $A$ to $B$, where $B$ is on the ground and $OB = 64\,\text{m}$, as shown in the diagram. By modelling the motion of the ball as that of a particle moving freely under gravity,

**a** find the value of $p$ and the value of $q$ **(5 marks)**

**b** find the initial speed of the ball **(2 marks)**

**c** find the exact value of $\tan\alpha$ **(1 mark)**

**d** find the length of time for which the cricket ball is at least 5 m above the ground. **(6 marks)**

**e** State an additional physical factor which may be taken into account in a refinement of the above model to make it more realistic. **(1 mark)**

E/P 6 In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively.

A clockwork train is moving on a flat, horizontal floor. At time $t = 0$, the train is at a fixed point $O$ and is moving with velocity $3\mathbf{i} + 13\mathbf{j}\,\text{m s}^{-1}$. The velocity of the train at time $t$ seconds is $\mathbf{v}\,\text{m s}^{-1}$, and its acceleration, $\mathbf{a}\,\text{m s}^{-2}$, is given by $\mathbf{a} = 2t\mathbf{i} + 3\mathbf{j}$.

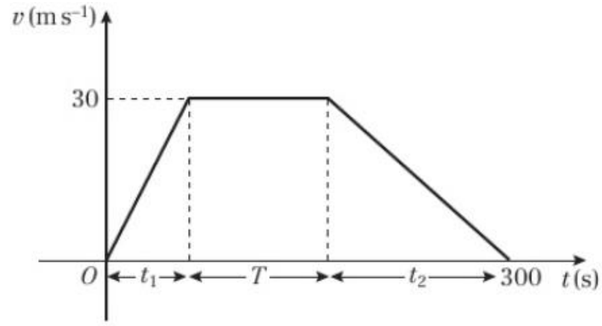
**a** Find $\mathbf{v}$ in terms of $t$. **(3 marks)**

**b** Find the value of $t$ when the train is moving in a north-east direction. **(3 marks)**

# Statistics & Mechanics 7. Kinematics

## Answers

1 **a**




**b** $\frac{30}{t_1} = 3x \Rightarrow t_1 = \frac{1}{x}, \frac{-30}{t_2} = -x \Rightarrow t_2 = \frac{30}{x}$

So $\frac{10}{x} + T + \frac{30}{x} = 300 \Rightarrow \frac{40}{x} + T = 300$

**c** $T = 100, x = 0.2$ **d** 3 km **e** 125 s

2 **a** Mass is not constant as fuel is used.
Gravity is not constant so weight not constant.
Thrust may not be constant.

**b** $v = (1.69 \times 10^{-7})\,t^4 - (1.33 \times 10^{-4})\,t^3 + 0.0525\,t^2 + 0.859\,t + 273\,\mathrm{m\,s^{-1}}$

**c** $v = 5990\,\mathrm{m\,s^{-1}}$

**d** 510 seconds (2 s.f.) after launch

3 **a** 0.65 s **b** 1.5 m **c** $23.8\,\mathrm{m\,s^{-1}}$

4 **a** Particle $P$: $x = 18t$, Particle $Q$: $x = 30 \cos \alpha \times t$
When particles collide: $18t = 30 \cos \alpha \times t \Rightarrow \cos\alpha = \frac{3}{5}$

**b** $\frac{4}{3}$ s

5 **a** $p = 16, q = 19.4$ **b** $25.1\,\mathrm{m\,s^{-1}}$

**c** $\frac{97}{80}$ **d** 3.50 s (3 s.f.)

**e** e.g. weight of the ball, air resistance

6 **a** $\mathbf{v} = (t^2 + 3)\mathbf{i} + (3t + 13)\mathbf{j}$ **b** 5 s

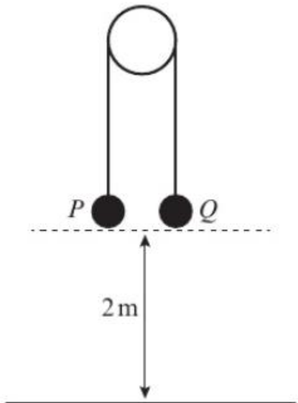
# Statistics & Mechanics 8. Forces

| | | | |
|---|---|---|---|
| **8**<br><br>**Forces and Newton's laws** | 8.1 | **Understand the concept of a force; understand and use Newton's first law.** | **Normal reaction, tension, thrust or compression, resistance.** |
| | 8.2 | **Understand and use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors);** extend to situations where forces need to be resolved (restricted to 2 dimensions). | **Problems will involve motion in a straight line with constant acceleration in scalar form, where the forces act either parallel or perpendicular to the motion.**<br><br>Extend to problems where forces need to be resolved, e.g. a particle moving on an inclined plane.<br><br>**Problems may involve motion in a straight line with constant acceleration in vector form, where the forces are given in $\mathbf{i}-\mathbf{j}$ form or as column vectors.** |
| | 8.3 | **Understand and use weight and motion in a straight line under gravity; gravitational acceleration, $g$, and its value in S.I. units to varying degrees of accuracy.** | **The default value of $g$ will be $9.8\text{ m s}^{-2}$ but some questions may specify another value, e.g. $g = 10\text{ m s}^{-2}$**<br><br>**The inverse square law for gravitation is not required and $g$ may be assumed to be constant, but students should be aware that $g$ is not a universal constant but depends on location.** |
| | 8.4 | **Understand and use Newton's third law; equilibrium of forces on a particle and motion in a straight line; application to problems involving smooth pulleys and connected particles;** resolving forces in 2 dimensions; equilibrium of a particle under coplanar forces. | **Problems may be set where forces need to be resolved (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors).**<br><br>**Connected particle problems could include problems with particles in contact e.g. lift problems.**<br><br>Problems may be set where forces need to be resolved, e.g. at least one of the particles is moving on an inclined plane. |
| | 8.5 | Understand and use addition of forces; resultant forces; dynamics for motion in a plane. | Students may be required to resolve a vector into two components or use a vector diagram, e.g. problems involving two or more forces, given in magnitude-direction form. |

| | | | |
|---|---|---|---|
| **8**<br><br>**Forces and Newton's laws**<br><br>*continued* | 8.6 | Understand and use the $F \leq \mu R$ model for friction; coefficient of friction; motion of a body on a rough surface; limiting friction and statics. | An understanding of $F = \mu R$ when a particle is moving.<br><br>An understanding of $F \leq \mu R$ in a situation of equilibrium. |
| **9**<br><br>**Moments** | 9.1 | Understand and use moments in simple static contexts. | Equilibrium of rigid bodies.<br><br>Problems involving parallel and non-parallel coplanar forces, e.g. ladder problems. |

# Statistics & Mechanics 8. Forces

E/P 1 Two particles $P$ and $Q$ have masses 0.5 kg and 0.4 kg respectively. The particles are attached to the ends of a light inextensible string. The string passes over a small smooth pulley which is fixed above a horizontal floor. Both particles are held, with the string taut, at a height of 2 m above the floor, as shown. The particles are released from rest and in the subsequent motion $Q$ does not reach the pulley.




**a** **i** Write down an equation of motion for $P$. **(2 marks)**

**ii** Write down an equation of motion for $Q$. **(2 marks)**

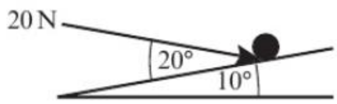
**b** Find the tension in the string immediately after the particles are released. **(2 marks)**

**c** Find the acceleration of $A$ immediately after the particles are released. **(2 marks)**

When the particles have been moving for 0.2 s, the string breaks.

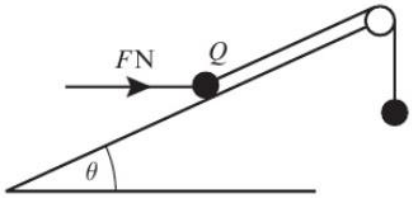
**d** Find the further time that elapses until $Q$ hits the floor. **(9 marks)**

E 2 A particle of mass 5 kg sits on a smooth slope that is inclined at 10° to the horizontal. A force of 20 N acts on the particle at an angle of 20° to the plane, as shown in the diagram. Find the acceleration of the particle. **(5 marks)**




E/P 3 A particle of mass 0.5 kg is being pulled up a rough slope that is angled at 30° to the horizontal by a force of 5 N. The force acts at an angle of 30° above the slope. Given that the coefficient of friction is 0.1, calculate the acceleration of the particle. **(7 marks)**

E/P 4 A particle $Q$ of mass 5 kg rests in equilibrium on a smooth inclined plane. The plane makes an angle $\theta$ with the horizontal, where $\tan\theta = \frac{3}{4}$.




$Q$ is attached to one end of a light inextensible string which passes over a smooth pulley as shown. The other end of the string is attached to a particle of mass 2 kg.

The particle $Q$ is also acted upon by a force of magnitude $F$ N acting horizontally, as shown in the diagram.

Find the magnitude of:

**a** the force $F$ **(5 marks)**

**b** the normal reaction between particle $Q$ and the plane. **(3 marks)**

The plane is now assumed to be rough.

**c** State, with a reason, which of the following statements is true:

1. $F$ will be larger 2. $F$ will be smaller 3. $F$ could be either larger or smaller. **(2 marks)**

E 5 A sledge of mass 50 kg sits on a snowy hill that is angled at 40° to the horizontal. The sledge is held in place by a rope that is angled at 30° above the line of greatest slope of the hill.

**a** By modelling the sledge as a particle, the hill as a smooth slope and the rope as a light inextensible string, work out the tension in the rope. **(4 marks)**

**b** Give one criticism of this model. **(1 mark)**

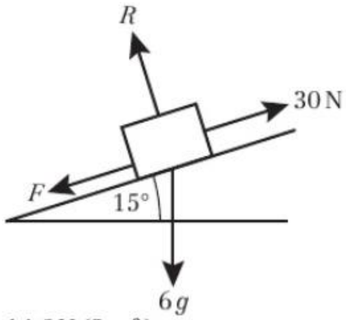
E/P 6 A trailer of mass 20 kg sits at rest on a rough horizontal plane. A force of 20 N pulls the trailer at an angle of 30° above the horizontal. Given that the trailer is in limiting equilibrium, work out the value of the coefficient of friction. **(6 marks)**

# Statistics & Mechanics 8. Forces

## Answers

1 **a** **i** $0.5g - T = 0.5a$ **ii** $T - 0.4g = 0.4a$
**b** $\frac{4}{9}g$ N **c** $\frac{1}{9}g\,\text{m s}^{-2}$ **d** 0.66 s (2 s.f.)

2 **a**




**b** 14.8 N (3 s.f.)

3 $3.41\,\text{ms}^{-2}$ (3 s.f.)

4 **a** 12.25 N **b** 46.6 N (3 s.f.)
**c** $F$ will be smaller as friction is acting up the slope.

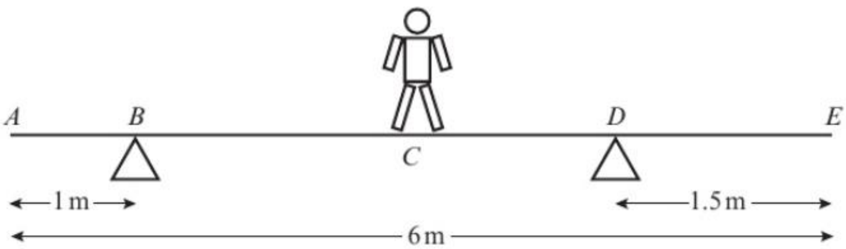
5 104 N, 64.5 N, 0.620

6 $\mu = \frac{5\sqrt{3}}{93}$

# Statistics & Mechanics 9. Moments

| | | | |
|---|---|---|---|
| 9<br><br>**Moments** | 9.1 | Understand and use moments in simple static contexts. | Equilibrium of rigid bodies.<br><br>Problems involving parallel and non-parallel coplanar forces, e.g. ladder problems. |

E 1 A plank $AE$, of length 6 m and weight 100 N, rests in a horizontal position on supports at $B$ and $D$, where $AB = 1$ m and $DE = 1.5$ m. A child of weight 145 N stands at $C$, the midpoint of $AE$, as shown in the diagram. The child is modelled as a particle and the plank as a uniform rod. The child and the plank are in equilibrium. Calculate:



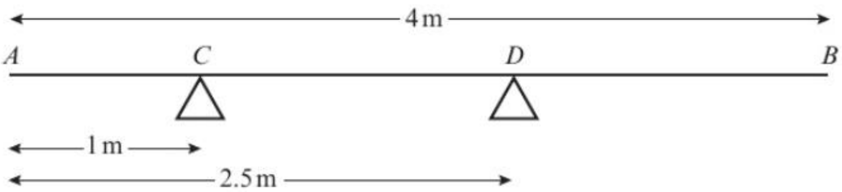

**a** the magnitude of the force exerted by the support on the plank at $B$ **(3 marks)**

**b** the magnitude of the force exerted by the support on the plank at $D$. **(2 marks)**

The child now stands at a different point $F$ on the plank. The plank is in equilibrium and on the point of tilting about $D$.

**c** Calculate the distance $DF$. **(4 marks)**

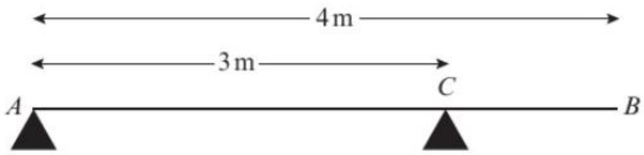
E/P 2 A uniform rod $AB$ has length 4 m and weight 150 N. The rod rests in equilibrium in a horizontal position, smoothly supported at points $C$ and $D$, where $AC = 1$ m and $AD = 2.5$ m as shown in the diagram. A particle of weight $W$ N is attached to the rod at a point $E$ where $AE = x$ metres. The rod remains in equilibrium and the magnitude of the reaction at $C$ is now equal to the magnitude of the reaction at $D$.




**a** Show that $W = \dfrac{150}{7 - 4x}$ **(6 marks)**

**b** Hence deduce the range of possible values of $x$. **(3 marks)**

3 A uniform plank $AB$ has mass 40 kg and length 4 m. It is supported in a horizontal position by two smooth pivots. One pivot is at the end $A$ and the other is at the point $C$ where $AC = 3$ m, as shown in the diagram.




A man of mass 80 kg stands on the plank which remains in equilibrium. The magnitude of the reaction at $A$ is twice the magnitude of the reaction at $C$. The magnitude of the reaction at $C$ is $R$ N. The plank is modelled as a rod and the man is modelled as a particle.

**a** Find the value of $R$. **(2 marks)**

**b** Find the distance of the man from $A$. **(3 marks)**

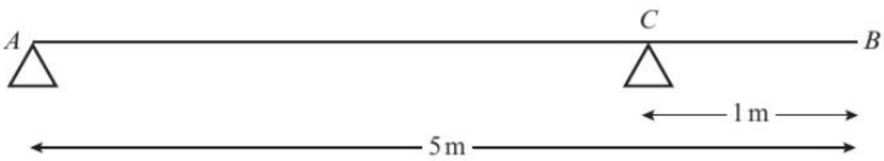
**c** State how you have used the modelling assumption that:

**i** the plank is uniform

**ii** the plank is a rod

**iii** the man is a particle. **(3 marks)**

4 A beam $AB$ has weight 200 N and length 5 m. The beam rests in equilibrium in a horizontal position on two smooth supports.



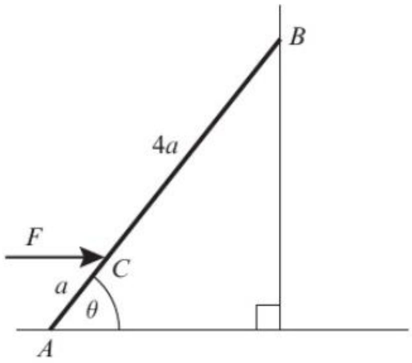

One support is at end $A$ and the other is at a point $C$ on the beam, where $BC = 1$ m, as shown in the diagram. The beam is modelled as a uniform rod.

**a** Find the reaction on the beam at $C$. **(4 marks)**

A woman of weight 500 N stands on the beam at the point $D$. The beam remains in equilibrium. The reactions on the beam at $A$ and $C$ are now equal.

**b** Find the distance $AD$. **(5 marks)**

E/P 5 A uniform ladder $AB$ has one end $A$ on smooth horizontal ground. The other end $B$ rests against a smooth vertical wall. The ladder is modelled as a uniform rod of mass $m$ and length $5a$. The ladder is kept in equilibrium by a horizontal force $F$ acting at a point $C$ of the ladder where $AC = a$. The force $F$ and the ladder lie in a vertical plane perpendicular to the wall. The ladder is inclined to the horizontal at an angle $\theta$, where $\tan\theta = \frac{9}{5}$, as shown in the diagram.




Show that $F = \frac{25mg}{72}$. **(8 marks)**

E/P 6 A uniform ladder, of weight $W$ and length 5 m, has one end on rough horizontal ground and the other touching a smooth vertical wall. The coefficient of friction between the ladder and the ground is 0.3.

The top of the ladder touches the wall at a point 4 m vertically above the level of the ground.

**a** Show that the ladder cannot rest in equilibrium in this position. **(6 marks)**

In order to enable the ladder to rest in equilibrium in the position described above, a brick is attached to the bottom of the ladder.

Assuming that this brick is at the lowest point of the ladder, but not touching the ground,

**b** show that the horizontal frictional force exerted by the ladder on the ground is independent of the mass of the brick **(4 marks)**

**c** find, in terms of $W$ and $g$, the smallest mass of the brick for which the ladder will rest in equilibrium. **(3 marks)**

# Statistics & Mechanics 9. Moments

## Answers

1 **a** 105 N **b** 140 N
**c** 1.03 m to the right of $D$

2 **a** R(↑) gives reaction at $C$ = reaction at $D = \frac{150 + W}{2}$
M($C$): $(1 \times 150) + W(x - 1) = 1.5\left(\frac{150 + W}{2}\right)$
$150 + Wx - W = 112.5 + 0.75W$
$37.5 = 1.75W - Wx \Rightarrow 150 = 7W - 4Wx$
$W = \frac{150}{7 - 4x}$
**b** $0 \leqslant x < \frac{7}{4}$

3 **a** 40 g **b** $x = \frac{1}{2}$
**c** **i** The weight acts at the centre of the plank.
**ii** The plank remains straight.
**iii** The man's weight acts at a single point.

4 **a** 125 N **b** 1.8 m

5 R(→): $F = N$
Taking moments about $A$
$Fa\sin\theta + \frac{5}{2}mga\cos\theta = 5aN\sin\theta$
$\frac{5}{2}mg\cos\theta = 4F\sin\theta$
$\frac{5}{8}mg = F\tan\theta$

6 **a** Taking moments about point where ladder touches the ground
R(↑): $R = W$, R(→): $N = 0.3R$
$1.5W = 1.2W$. This cannot be true so the ladder cannot rest in this position.
**b** R(→): $F = N$
Taking moments about point where ladder touches the ground $1.5W = 4N$, $F = N = \frac{3W}{8}$
**c** $\frac{W}{4g}$

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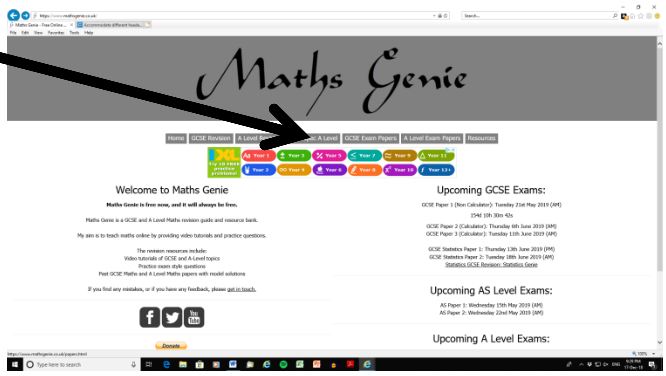


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| Topic | | Videos | Exam Questions |
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| Equations and Inequalities | 2 | Videos | Quadratics Inequalities and Simultaneous Equations<br>The Discriminant |
| Sketching Curves | | Videos | Sketching and Transforming Curves |
| Equations of Straight Lines | | Videos | The Equation of a Line |
| Circles | | Videos | The Equation of a Circle |
| Trigonometry | 5 | Videos | Solving Trigonometric Equations<br>Sine Rule, Cosine Rule, Area of Any Triangle |
| Exponentials and Logarithms | 6 | Videos | Exponentials and Logarithms |
| Differentiation | 7 | Videos | Differentiation from First Principles<br>Differentiation |
| Integration | 8 | Videos | Integration |
| Vectors | 10 | Videos | Vectors |
| Proof | 1 | Videos | Proof |

...continued

# Extra Questions

## AS Mechanics and Statistics

| Topic | | Videos | Exam Questions |
|---|---|---|---|
| Data Presentation and Interpretation | 12 | Videos | Histograms<br>Box Plots<br>Interpolation and Standard Deviation<br>Correlation and Regression |
| Probability and Statistical Distributions | 13 | Videos | Probability<br>Discrete Random Variables |
| Statistical Sampling and Hypothesis Testing | 12/15 | Videos | Sampling<br>Binomial Hypothesis Testing |
| Kinematics | 16 | Videos | SUVAT<br>Velocity Time Graphs<br>Variable Acceleration |
| Forces | 17 | Videos | 2D Vectors<br>F = ma |

## A Level Pure Mathematics

| Topic | | Videos | Exam Questions |
|---|---|---|---|
| Functions | 2 | Videos | Functions<br>Transforming Graphs |
| Partial Fractions | 2 | Videos | Partial Fractions |
| Parametric Equations | 3 | Videos | Parametric Equations |
| Sequences and Series | 4 | Videos | Recurrence Relations<br>Arithmetic Sequences and Series<br>Geometric Sequences and Series<br>The Binomial Expansion |
| Trigonometry | 5 | Videos | Radians<br>Small Angle Approximations<br>Sec, Cosec and Cot<br>Trig Identities<br>Addition and Double Angle Formulae<br>R Formulae |
| Differentiation | 7 | Videos | The Chain Rule<br>The Product Rule<br>The Quotient Rule<br>Trigonometric Differentiation<br>Implicit Differentiation<br>Cos and Sin from First Principles |
| Integration | 8 | Videos | Trigonometric Integration<br>Exponential Integration<br>Integration by Substitution<br>Integration by Parts<br>Parametric Integration<br>Differential Equations |
| Numerical Methods | 9 | Videos | Iteration<br>Newton-Raphson<br>The Trapezium Rule |
| 3D Vectors | 10 | Videos | 3D Vectors |
| Proof | 2 | Videos | Proof by Contradiction |

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# Extra Questions

A Level Mechanics and Statistics

| Topic | | Videos | Exam Questions |
|---|---|---|---|
| Probability | 13 | Videos | Probability |
| Statistical Distributions | 14 | Videos | The Normal Distribution<br>Using the Normal Distribution to approximate the Binomial |
| Statistical Hypothesis Testing | 15 | Videos | Correlation Hypothesis Testing<br>Mean of Normal Distribution Hypothesis Testing<br>Non Linear Regression |
| Forces | 17 | Videos | Resolving Forces<br>Resolving Forces 2<br>Connected Particles |
| Kinematics | 16 | Videos | Kinematics with Vectors<br>Kinematics with Calculus<br>Projectiles |
| Moments | 18 | Videos | Moments<br>Statics of Rigid Bodies |