FP2 PRACTICE PAPER 5

1. (*a*) Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = x.$$

(5)

Given that $y = 1$ at $x = 0$,

(*b*) find the exact values of the coordinates of the minimum point of the particular solution curve,

(4)

(*c*) draw a sketch of this particular solution curve.

(2)

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2. (*a*) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2e^{-t}.$$

(6)

(*b*) Find the particular solution that satisfies $y = 1$ and $\frac{dy}{dt} = 1$ at $t = 0$.

(6)

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3. **Figure 1**

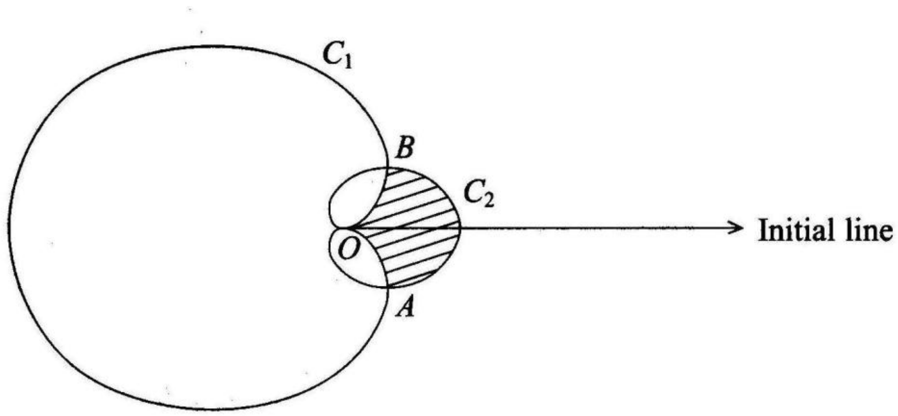

Figure 1 is a sketch of the two curves C_1 and C_2 with polar equations

$$C_1 : r = 3a(1 - \cos\theta), \qquad -\pi \le \theta < \pi$$

and $$C_2 : r = a(1 + \cos\theta), \qquad -\pi \le \theta < \pi.$$

The curves meet at the pole O, and at the points A and B.

(*a*) Find, in terms of a, the polar coordinates of the points A and B.

(4)

(b) Show that the length of the line AB is $\frac{3\sqrt{3}}{2}a$.

(2)

The region inside C_2 and outside C_1 is shown shaded in Fig. 1.

(c) Find, in terms of a, the area of this region.

(7)

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

(d) calculate the area of this badge, giving your answer to three significant figures.

(3)

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4. The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, \qquad z \neq -\mathrm{i}.$$

(a) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle
$|w| = 1$ in the w-plane.

(4)

(b) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane.

(6)

(c) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane.

(2)

(d) Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane.

(2)

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5. (a) Show that

$$\frac{r^3 - r + 1}{r(r+1)} \equiv r - 1 + \frac{1}{r} - \frac{1}{r+1}, \quad \text{for } r \neq 0, -1.$$

(3)

(b) Find $\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)}$, expressing your answer as a single fraction in its simplest form.

(6)

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6. De Moivre's theorem states that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \text{ for } n \in \mathbb{R}.$$

(*a*) Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$.

(5)

(*b*) Show that

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$

(5)

(*c*) Hence show that $2 \cos \frac{\pi}{10}$ is a root of the equation $x^4 - 5x^2 + 5 = 0$.

(3)

TOTAL MARKS: 75