C4, IYGB, PAPER Z

1. a)

$$f(x) = \frac{1}{(2-3x)^3} = (2-3x)^{-3}$$

$$f(x) = (2-3x)^{-3} = 2^{-3}\left(1-\tfrac{3}{2}x\right)^{-3} = \tfrac{1}{8}\left(1-\tfrac{3}{2}x\right)^{-3}$$

$$= \tfrac{1}{8}\left[1 + \frac{-3}{1}\left(-\tfrac{3}{2}x\right)^1 + \frac{-3(-4)}{1\times 2}\left(\tfrac{3}{2}x\right)^2 + O(x^3)\right]$$

$$= \tfrac{1}{8}\left[1 + \tfrac{9}{2}x + \tfrac{27}{2}x^2 + O(x^3)\right]$$

$$= \tfrac{1}{8} + \tfrac{9}{16}x + \tfrac{27}{16}x^2 + O(x^3)$$

b)

$$\frac{2+px}{(2-3x)^3} = (2+px)(2-3x)^3$$

$$= (2+px)\left(\tfrac{1}{8} + \tfrac{9}{16}x + \tfrac{27}{16}x^2 + O(x^3)\right)$$

$$= \tfrac{1}{4} + \tfrac{9}{8}x + \tfrac{27}{8}x^2 + O(x^3)$$

$$\tfrac{1}{8}px + \tfrac{9}{16}px^2 + O(x^3)$$

$$= \tfrac{1}{4} + \underbrace{\left(\tfrac{9}{8}+\tfrac{1}{8}p\right)}_{\frac{1}{8}}x + \underbrace{\left(\tfrac{27}{8} + \tfrac{9}{16}p\right)}_{q}x^2 + O(x^3)$$

- $\tfrac{9}{8} + \tfrac{1}{8}p = \tfrac{1}{8}$

  $\tfrac{1}{8}p = -1$

  $p = -8$

- $\tfrac{27}{8} + \tfrac{9}{16}p = q$

  $q = \tfrac{27}{8} + \tfrac{9}{16}\times(-8)$

  $q = \tfrac{27}{8} - \tfrac{9}{2}$

  $q = -\tfrac{9}{8}$

C4, IYGB, PAPER Z

2. a) $(k,k) \Longrightarrow k^3 + k^3 = 8k \times k$

$$2k^3 = 8k^2$$
$$2k = 8 \quad \Big) \; k \neq 0.$$
$$k = 4$$

b) $x^3 + y^3 = 8xy$

• Diff w.r.t x

$$3x^2 + 3y^2\frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$$

• AT $(4,4)$

$$3\times4^2 + 3\times4^2\left.\frac{dy}{dx}\right|_{(4,4)} = 8\times4 + 8\times4\times\left.\frac{dy}{dx}\right|_{(4,4)}$$

$$48 + 48\left.\frac{dy}{dx}\right|_{(4,4)} = 32 + 32\left.\frac{dy}{dx}\right|_{(4,4)}$$

$$16\left.\frac{dy}{dx}\right|_{(4,4)} = -16$$

$$\left.\frac{dy}{dx}\right|_{(4,4)} = -1$$

3. a) $\frac{dV}{dt} = 300$, GIVEN

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\Rightarrow \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \times 300$$

$$\Rightarrow \frac{dr}{dt} = \frac{75}{\pi r^2}$$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{r=15} = \frac{75}{\pi \times 15^2} = \frac{1}{3\pi} = 0.1061 \text{ cm s}^{-1}$$

b) • $t=0 \quad V=0$

$t=1 \quad V=300$

$t=2 \quad V=600$

$t=3 \quad V=900$

$\vdots$

$t=10 \quad V=3000$

• NOW

$$V = \frac{4}{3}\pi r^3$$
$$3000 = \frac{4}{3}\pi r^3$$
$$r^3 = \frac{9000}{4\pi}$$
$$r \approx 8.94700\ldots$$

• THUS

$$\left.\frac{dr}{dt}\right|_{t=10} = \left.\frac{dr}{dt}\right|_{r=8.947\ldots}$$

$$= \frac{75}{\pi \times 8.947\ldots^2}$$

$$\approx 0.298 \text{ cms}^{-1}$$

4. AREA = 10

$$\Rightarrow \int_1^a \frac{5}{\sqrt{5x-4}}\,dx = 10$$

$$\Rightarrow \int_1^a 5(5x-4)^{-\frac{1}{2}}\,dx = 10$$

$$\Rightarrow \left[2(5x-4)^{\frac{1}{2}}\right]_1^a = 10$$

$$\Rightarrow 2(5a-4)^{\frac{1}{2}} - 2 = 10$$

$$\Rightarrow (5a-4)^{\frac{1}{2}} = 6$$

$$\Rightarrow 5a-4 = 36$$

$$\Rightarrow 5a = 40$$

$$\Rightarrow a = 8$$

NOW VOLUME

$$\Rightarrow V = \pi\int_{x_1}^{x_2} \left(y(x)\right)^2\,dx$$

$$\Rightarrow V = \pi\int_1^8 \left(\frac{5}{\sqrt{5x-4}}\right)^2\,dx$$

$$\Rightarrow V = \pi\int_1^8 \frac{25}{5x-4}\,dx$$

$$\Rightarrow V = \pi\left[5\ln|5x-4|\right]_1^8$$

$$\Rightarrow V = 5\pi\left[\ln 36 - \cancel{\ln 1}\right]$$

$$\Rightarrow V = 5\pi \ln 36$$

$$\Rightarrow V = 5\pi(2\ln 6)$$

$$\Rightarrow V = 10\pi \ln 6$$

5. a)

$$x = 6(2\theta - \sin 2\theta) \qquad y = 6(1-\cos 2\theta)$$

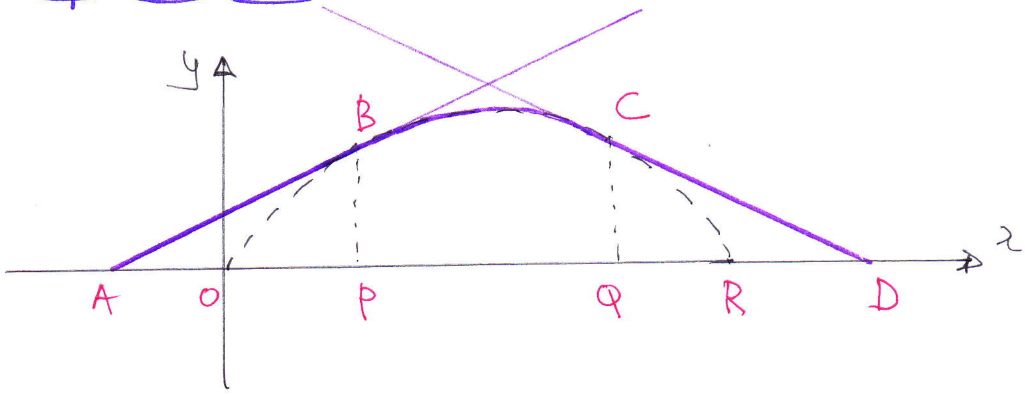
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{6}(2\sin 2\theta)}{\cancel{6}(2-2\cos 2\theta)} = \frac{2\sin 2\theta}{2-2\cos 2\theta}$$

$$= \frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cancel{1}-(\cancel{1}-2\sin^2\theta)} = \frac{\cancel{2\sin\theta}\cos\theta}{\cancel{2\sin^2\theta}}$$

$$= \frac{\cos\theta}{\sin\theta} = \cot\theta$$

AS REQUIRED

b) I)



- AT B, $\theta = \frac{\pi}{3}$

$$x = 6\left(2\times\frac{\pi}{3} - \sin\frac{2\pi}{3}\right) = 6\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 4\pi - 3\sqrt{3}$$

$$y = 6\left(1 - \cos\frac{2\pi}{3}\right) = 6\left(1+\frac{1}{2}\right) = 9$$

$$\frac{dy}{dx} = \cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

EQUATION OF TANGENT

$$y - y_0 = m(x - x_0)$$

$$y - 9 = \frac{1}{\sqrt{3}}\left(x - (4\pi - 3\sqrt{3})\right)$$

WHEN $y=0$

$$-9 = \frac{1}{\sqrt{3}}(x - 4\pi + 3\sqrt{3})$$

$$-9\sqrt{3} = x - 4\pi + 3\sqrt{3}$$

$$x = 4\pi - 12\sqrt{3}$$

AS REQUIRED

II) $|AP| = \underbrace{(4\pi - 3\sqrt{3})}_{\text{SAME } x \text{ AS B}} - \underbrace{(4\pi - 12\sqrt{3})}_{\text{POINT A}} = 9\sqrt{3}$

III) $\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\,\frac{dx}{d\theta}\,d\theta$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 6(1-\cos 2\theta) \times 6(2 - 2\cos 2\theta)\,d\theta$$

$$= 36\int_{\pi/3}^{2\pi/3} 2(1-\cos 2\theta)^2 \, d\theta = 36\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 2(1-2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= 36\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 2\left[1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= 36\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 3 - 4\cos 2\theta + \cos 4\theta \, d\theta$$

AS REQUIRED

c) INTEGRATE

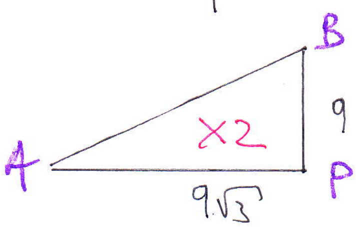
$$36\left[3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$36\left[\left(2\pi + \sqrt{3} + \frac{1}{8}\sqrt{3}\right) - \left(\pi - \sqrt{3} - \frac{1}{8}\sqrt{3}\right)\right]$$

$$36\left[\pi + 2\sqrt{3} + \frac{1}{4}\sqrt{3}\right]$$

$$9\left[4\pi + 9\sqrt{3}\right]$$

$+$



$$\leftarrow \left[\frac{1}{2} \times 9 \times 9\sqrt{3}\right] \times 2$$

$$\therefore \; 9(4\pi + 9\sqrt{3}) + 81\sqrt{3}$$

$$36\pi + 162\sqrt{3}$$

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6. a)

$$r_1 = (5,3,6) + \lambda(2,1,2) = (2\lambda+5,\ \lambda+3,\ 2\lambda+6)$$
$$r_2 = (-1,5,a) + \mu(1,-2,-2) = (\mu-1,\ 5-2\mu,\ a-2\mu)$$

EQUATE i & j

(i): $2\lambda+5 = \mu-1$
(j): $\lambda+3 = 5-2\mu$ $\Rightarrow \boxed{\lambda = 2-2\mu}$ $\Rightarrow$ $2(2-2\mu)+5 = \mu-1$

$4-4\mu+5 = \mu-1$

$10 = 5\mu$

$\boxed{\mu = 2}$

& $\boxed{\lambda = -2}$

$\therefore A(2\times(-2)+5,\ -2+3,\ 2(-2)+6)$

$A(1,1,2)$

& FROM k: $2\lambda+6 = a-2\mu$

$-4+6 = a-4$

$a = 6$

b) BY INSPECTION OF $x=11 \Rightarrow \lambda = 3 \quad \therefore p = 6$

BY INSPECTION OF $y=-9 \Rightarrow 5-2\mu = -9$

$14 = 2\mu$

$\mu = 7 \quad \therefore q = 6$

c) $P(11,6,12)$

$Q(6,-9,-8)$ $\quad \therefore M\left(\frac{17}{2}, -\frac{3}{2}, 2\right)$

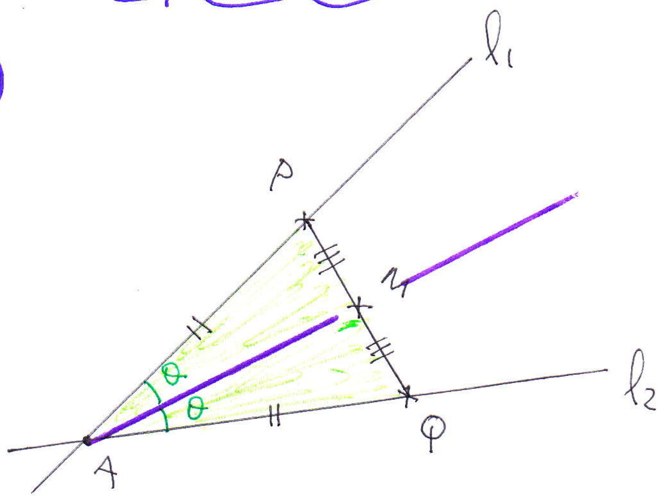
d) $|AP| = |\underline{p}-\underline{a}| = |(11,6,12)-(1,1,2)| = |10,5,10| = \sqrt{100+25+100}$

$= \sqrt{225} = 15$

$|AQ| = |\underline{q}-\underline{a}| = |(6,-9,-8)-(1,1,2)| = |5,-10,-10| = \sqrt{25+100+100}$

$= \sqrt{225} = 15 \qquad \therefore |AP| = |AQ|$

e)



- $\vec{AM} = \underline{m} - \underline{a}$

$= \left(\frac{17}{2}, -\frac{3}{2}, 2\right) - (1,1,2)$

$= \left(\frac{15}{2}, -\frac{5}{2}, 0\right)$

- USE THIS AS DIRECTION VECTOR

$\left(\frac{15}{2}, -\frac{5}{2}, 0\right)$

SCALES TO

$(3, -1, 0)$

- $\underline{r}_3 = (1,1,2) + t(3,-1,0)$

$\underline{r}_3 = (3t+1, 1-t, 2)$

7. a)

$$\frac{dP}{dt} = +kP(3-P)$$

RATE OF GROWTH

PROPORTIONALITY CONSTANT

POPULATION

DIFFERENCE OF POPULATION FROM 3 MILLION

P = population in millions

t = time in years

$t=0, \quad P=1$

- SEPARATE VARIABLES

$$\Rightarrow \frac{1}{P(3-P)}\,dP = k\,dt$$

$$\Rightarrow \int \frac{1}{P(3-P)}\,dP = \int k\,dt$$

BY PARTIAL FRACTIONS

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$$\frac{1}{P(3-P)} \equiv \frac{A}{P} + \frac{B}{3-P}$$

$$\boxed{1 \equiv A(3-P) + BP}$$

when $P=0$ $\quad 1 = 3A \implies A = \frac{1}{3}$

$P=3$ $\quad 1 = 3B \implies B = \frac{1}{3}$

$$\dots \implies \int \frac{1/3}{P} + \frac{1/3}{3-P}\, dP = \int k\, dt$$

$$\implies \int \frac{1}{P} + \frac{1}{3-P}\, dP = \int a\, dt$$

×3 (k IS JUST A CONSTANT, $a = 3k$)

$$\implies \ln|P| - \ln|3-P| = at + C$$

$$\implies \ln\left|\frac{P}{3-P}\right| = at + C$$

$$\implies \frac{P}{3-P} = e^{at+C}$$

$$\implies \frac{P}{3-P} = e^{at} \times e^{C}$$

$$\implies \frac{P}{3-P} = Ae^{at} \quad (A = e^{C})$$

APPLY $t=0$ $P=1$ $\quad \frac{1}{3-1} = Ae^{0}$

$$\boxed{A = \frac{1}{2}}$$

$$\implies \frac{P}{3-P} = \frac{1}{2}e^{at}$$

$$\implies \frac{2P}{3-P} = e^{at}$$

AS REQUIRED

C4, IYGB, PAPER Z

b) $t=10 \quad P=2$

$$\frac{2\times 2}{3-2} = e^{10a}$$

$$4 = e^{10a}$$

$$\ln 4 = 10a$$

$$2\ln 2 = 10a$$

$$a = \frac{1}{5}\ln 2$$

d) $$\boxed{\frac{2P}{3-P} = e^{\left(\frac{1}{5}\ln 2\right)t}}$$

$$\Rightarrow \frac{2P}{3-P} = \left(e^{\ln 2}\right)^{\frac{1}{5}t}$$

$$\Rightarrow \frac{2P}{3-P} = 2^{\frac{1}{5}t}$$

$$\Rightarrow \frac{3-P}{2P} = 2^{-\frac{1}{5}t}$$

$$\Rightarrow \frac{3}{2P} - \frac{1}{2} = 2^{-\frac{1}{5}t}$$

$$\Rightarrow \frac{3}{2P} = \frac{1}{2} + 2^{-\frac{1}{5}t}$$

$$\Rightarrow \frac{2P}{3} = \frac{1}{\frac{1}{2}+2^{-\frac{1}{5}t}}$$

$$\Rightarrow P = \frac{\frac{3}{2}}{\frac{1}{2}+2^{-\frac{1}{5}t}}$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY

$$\Rightarrow P = \frac{3}{1+2\times 2^{-\frac{1}{5}t}}$$

$$\Rightarrow P = \frac{3}{1+2^{1-\frac{1}{5}t}}$$

AS REQUIRED

ALTERNATIVE

$$\frac{2P}{3-P} = 2^{\frac{1}{5}t}$$

$$\Rightarrow 2P = (3-P)\times 2^{\frac{1}{5}t}$$

$$\Rightarrow 2P = 3\times 2^{\frac{1}{5}t} - P\times 2^{\frac{1}{5}t}$$

$$\Rightarrow 2P + P\times 2^{\frac{1}{5}t} = 3\times 2^{\frac{1}{5}t}$$

$$\Rightarrow P\left(2+2^{\frac{1}{5}t}\right) = 3\times 2^{\frac{1}{5}t}$$

$$\Rightarrow P = \frac{3\times 2^{\frac{1}{5}t}}{2+2^{\frac{1}{5}t}}$$

$$\Rightarrow P = \frac{3\times 2^{\frac{1}{5}t}\times 2^{-\frac{1}{5}t}}{2\times 2^{-\frac{1}{5}t} + 2^{\frac{1}{5}t}\times 2^{-\frac{1}{5}t}}$$

$$\Rightarrow P = \frac{3}{2\times 2^{-\frac{1}{5}t}+1}$$

$$\Rightarrow P = \frac{3}{2^{1-\frac{1}{5}t}+1}$$

AS BEFORE