C4, IYGB, PAPER Y

1. $\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx = \int \frac{\cos 2x}{\sin 2x} \times \frac{1}{\sin 2x}\,dx$

$\int \cot 2x \operatorname{cosec} 2x \,dx = -\frac{1}{2}\operatorname{cosec} 2x + C$

2. a) when $h = 0.5$ $\quad V = \frac{1}{3}\pi(0.5)^2(3-0.5)$

$V = \frac{5}{24}\pi$

$V \simeq 0.654$

b) $\frac{dV}{dt} = \frac{\pi}{24}$

$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi(2h-h^2)} \times \frac{\pi}{24}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{24(2h-h^2)}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{t=5} = \left.\frac{dh}{dt}\right|_{h=\frac{1}{2}} = \frac{1}{18}$

FROM PART (a).

when $h = 0.5$ $\quad V = \frac{5\pi}{24}$

$V = \frac{1}{3}\pi h^2(3-h)$

$V = \frac{1}{3}\pi(3h^2 - h^3)$

$\frac{dV}{dh} = \frac{1}{3}\pi(6h - 3h^2)$

$\frac{dV}{dh} = \pi(2h - h^2)$

$\frac{dh}{dV} = \frac{1}{\pi(2h-h^2)}$

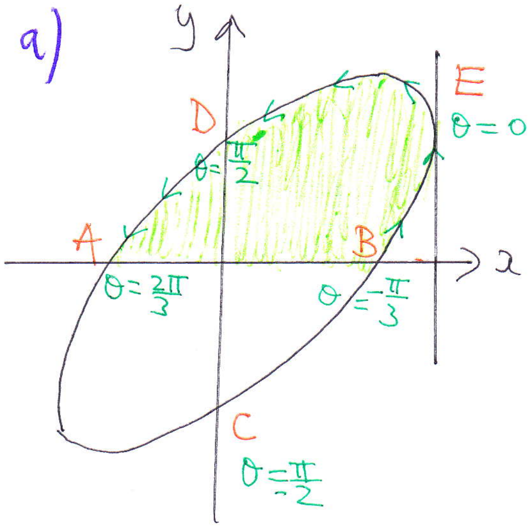
NOW IN 1 hour $\frac{\pi}{24}\,m^3$

2 hours $\frac{\pi}{24} \times 2$

$\vdots$

5 hours $\frac{5\pi}{24}$

3. a)



$$x = 2\cos\theta$$
$$y = 6\sin\left(\theta + \frac{\pi}{3}\right)$$
$$-\pi \le \theta < \pi$$

- $x = 0$

$\cos\theta = 0$

$\theta = \left\langle \begin{matrix} \pi/2 \\ -\pi/2 \end{matrix} \right.$ $\quad y = \left\langle \begin{matrix} 3 \\ -3 \end{matrix} \right.$ $\quad \therefore$ $C(0,-3)$, $D(0,3)$

- $y = 0$

$\sin(\theta + \pi/3) = 0$

$\left.\begin{matrix} \theta + \pi/3 = 0 \\ \theta + \pi/3 = \pi \end{matrix}\right\} \Rightarrow \theta = \left\langle \begin{matrix} -\pi/3 \\ 2\pi/3 \end{matrix} \right.$ $\quad x = \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right.$ $\quad A(-1,0)$, $B(1,0)$

b) $x = 2$ $\quad \left(\begin{matrix} x = 2\cos\theta \\ x_{MAX} = 2 \end{matrix} \right)$

c) $\cos\theta = 1$ $\quad \theta = 0$ (ONLY SOLUTION)

d) FIND THE DIRECTION OF THE CURVE FROM THE REFERENCE POINTS A – E (SEE DIAGRAM)

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$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

B

A

$$= \int_{\frac{2\pi}{3}}^{-\frac{\pi}{3}} 6\sin\left(\theta+\frac{\pi}{3}\right)(-2\sin\theta)\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} 12\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\,d\theta$$

$$= \int_{-\pi/3}^{\frac{2\pi}{3}} 12\sin\theta\left[\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\right]\,d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} 12\sin\theta\left[\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right]\,d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} 6\sin^2\theta + 6\sqrt{3}\sin\theta\cos\theta\,d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} 6\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right) + 3\sqrt{3}\left(2\sin\theta\cos\theta\right)\,d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} 3 - 3\cos 2\theta + 3\sqrt{3}\sin 2\theta\,d\theta$$

AS REQUIRED

C4, IYGB, PAPER Y

e) $\dots = \left[3\theta - \frac{3}{2}\sin 2\theta - \frac{3}{2}\sqrt{3}\cos 2\theta\right]_{-\pi/3}^{2\pi/3}$

$= \left[2\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{3}{2}\sqrt{3}\left(-\frac{1}{2}\right)\right]$
$- \left[-\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{3}{2}\sqrt{3}\left(-\frac{1}{2}\right)\right]$

$= \left[2\pi + \cancel{\frac{3}{4}\sqrt{3}} + \cancel{\frac{3}{4}\sqrt{3}}\right] - \left[-\pi + \cancel{\frac{3}{4}\sqrt{3}} + \cancel{\frac{3}{4}\sqrt{3}}\right]$

$= 3\pi$

4. a) $\underline{r}_1 = (5,-2,1) + \lambda(2,0,1) = (2\lambda+5, -2, \lambda+1)$

b) $\underline{r}_2 = (0,4,3) + \mu(-1,2,1) = (-\mu, 2\mu+4, \mu+3)$

- EQUATE $\underline{j}$

$2\mu+4 = -2$
$2\mu = -6$
$\boxed{\mu = -3}$

EQUATE $\underline{i}$

$2\lambda+5 = -\mu$
$2\lambda+5 = 3$
$2\lambda = -2$
$\boxed{\lambda = -1}$

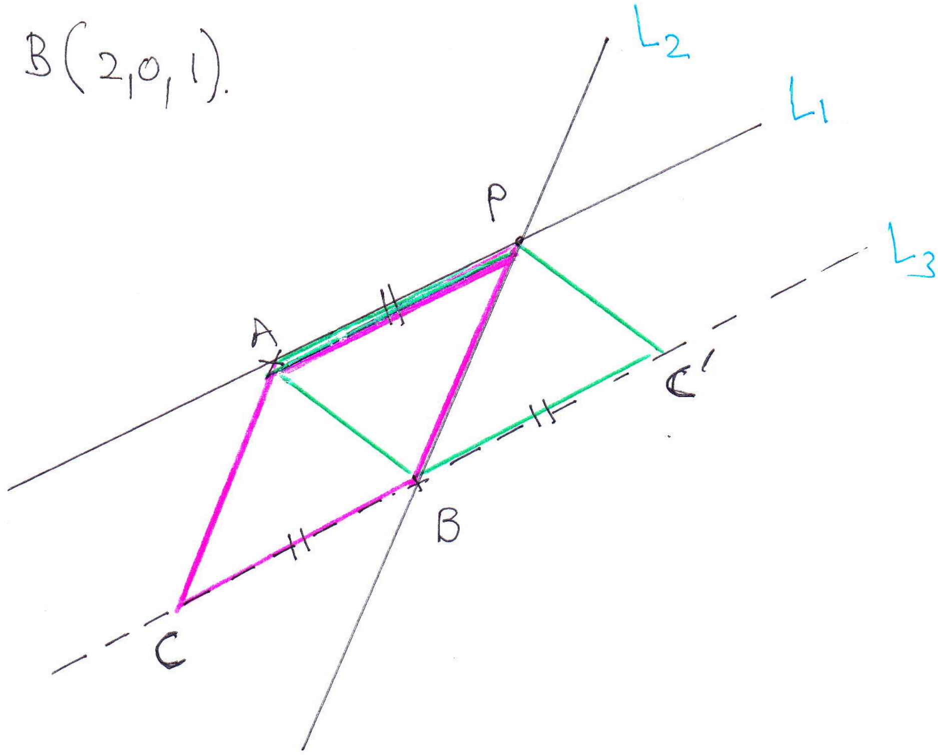
CHECK $\underline{k}$

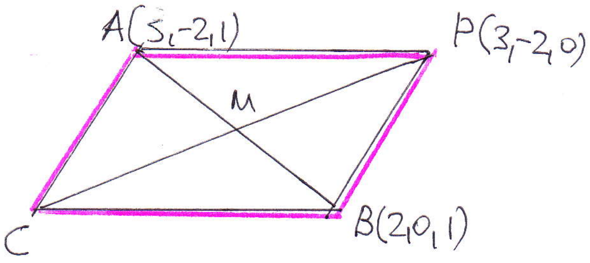
$\lambda+1 = -1+1 = 0$
$\mu+3 = -3+3 = 0$

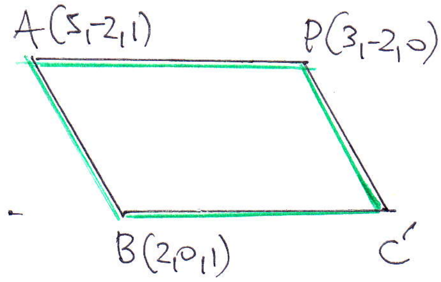
As all 3 components agree the lines intersect

USING $\lambda=-1$ GIVES $P(3,-2,0)$

c) $B(2,0,1)$.







- M IS THE MIDPOINT OF "AB"

  $M\left(\frac{7}{2},-1,1\right)$

- M MUST ALSO BE THE MIDPOINT OF "PC"

	P		M		C
x	3	$\xrightarrow{+0.5}$	$\frac{7}{2}$	$\xrightarrow{+0.5}$	4
y	−2	$\xrightarrow{+1}$	−1	$\xrightarrow{+1}$	0
z	0	$\xrightarrow{+1}$	1	$\xrightarrow{+1}$	2

- NOTING THAT $|CB| = |BC'|$

  B IS ALSO THE MIDPOINT OF "CC′"

	C		B		C′
x	4	$\xrightarrow{-2}$	2	$\xrightarrow{-2}$	0
y	0	$\xrightarrow{+0}$	0	$\xrightarrow{+0}$	0
z	2	$\xrightarrow{-1}$	1	$\xrightarrow{-1}$	0

$\therefore$ $(0,0,0)$ OR $(4,0,2)$

C4, IYGB, PAPER Y

5. a) $x^2+2x+y^3 = 63+xy$

Diff with respect to x

$\Rightarrow 2x+2+3y^2\frac{dy}{dx} = 0+(1\times y)+\left(x\times 1\times\frac{dy}{dx}\right)$

$\Rightarrow 2x+2+3y^2\frac{dy}{dx} = y+x\frac{dy}{dx}$

$\Rightarrow 3y^2\frac{dy}{dx}-x\frac{dy}{dx} = y-2x-2$

$\Rightarrow (3y^2-x)\frac{dy}{dx} = y-2x-2$

$\Rightarrow \frac{dy}{dx} = \frac{y-2x-2}{3y^2-x}$

b) FOR STATIONARY POINTS $\frac{dy}{dx}=0$

$y-2x-2=0$

$\boxed{y=2x+2}$

- BUT THESE POINTS MUST ALSO LIE ON THE CURVE

$x^2+2x+y^3 = 63+xy$

- SOLVING SIMULTANEOUSLY

$\Rightarrow x^2+2x+(2x+2)^3 = 63+x(2x+2)$

$\Rightarrow x^2+2x+\left[8x^3+3(2x)^2(2)+3(2x)2^2-8\right] = 63+2x^2+2x$

$(a\pm b)^3 = a^3\pm 3a^2b+3ab^2\pm b^3$

$\Rightarrow x^2+\cancel{2x}+8x^3+24x^2+24x+8 = 63+2x^2+\cancel{2x}$

$\Rightarrow 8x^3+23x^2+24x-55=0$

C4, IYGB, PAPER X

BY INSPECTION OR LONG DIVISION

$$\begin{array}{r}
8x^2+31x+55 \\
x-1 \,\big)\, 8x^3+23x^2+24x-55 \\
-8x^3+8x^2 \\
\hline
31x^2+24x-55 \\
-31x^2+31x \\
\hline
55x-55 \\
-55x+55 \\
\hline
\end{array}$$

$\therefore (x-1)(8x^2+31x+55)=0$

Δ $b^2-4ac = 31^2-4\times8\times55$

$961-1760<0$

$\therefore$ ONLY SOLUTION IS $x=1$

$y=2x+2=2\times1+2=4$ $\therefore (1,4)$

6. $\sqrt{\dfrac{1+ax}{4-x}} = (1+ax)^{\frac{1}{2}}(4-x)^{-\frac{1}{2}}$

$= (1+ax)^{\frac{1}{2}} \times 4^{-\frac{1}{2}}\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}$

$= (1+ax)^{\frac{1}{2}} \times \frac{1}{2}\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}$

EXPANDING

$f(x) = \frac{1}{2}\left[1+\frac{\frac{1}{2}}{1}(ax)^1+\frac{\frac{1}{2}(-\frac{1}{2})}{1\times2}(ax)^2+O(x^3)\right]\left[1+\frac{-\frac{1}{2}}{1}\left(-\frac{1}{4}x\right)^1+\frac{-\frac{1}{2}(-\frac{3}{2})}{1\times2}\left(-\frac{1}{4}x\right)^2+O(x^3)\right]$

$= \frac{1}{2}\left[1+\frac{1}{2}ax-\frac{1}{8}a^2x^2+O(x^3)\right]\left[1+\frac{1}{8}x+\frac{3}{128}x^2+O(x^3)\right]$

MULTIPLYING OUT ONLY THE TERMS WHICH PRODUCE x^2 WITHOUT FORGETTING THE $\frac{1}{2}$ AT THE FRONT.

$$\frac{1}{2}\left[\frac{3}{128}+\frac{1}{16}a-\frac{1}{8}a^2\right]=\frac{1}{64}$$

$$\frac{3}{128}+\frac{1}{16}a-\frac{1}{8}a^2=\frac{1}{32}$$

$$3+8a-16a^2=4$$

$$0=16a^2-8a+1$$

$$(4a-1)^2=0$$

$$a=\frac{1}{4}$$

7. a)

$$y=2\sin 2x+3\cos 2x$$

$$y^2=(2\sin 2x+3\cos 2x)^2$$

$$y^2=4\sin^2 2x+12\sin 2x\cos 2x+9\cos^2 2x$$

USING
$$\sin 2A=2\sin A\cos A$$
$$\cos^2 A=\frac{1}{2}+\frac{1}{2}\cos 2A$$
$$\sin^2 A=\frac{1}{2}-\frac{1}{2}\cos 2A$$

$$y^2=4\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)+6(2\sin 2x\cos 2x)+9\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)$$

$$y^2=2-2\cos 4x+6\sin 4x+\frac{9}{2}+\frac{9}{2}\cos 4x$$

$$y^2=\frac{13}{2}+\frac{5}{2}\cos 4x+6\sin 4x$$

$A=13/2$
$B=5/2$
$C=4$

b) $\text{AREA} = \int_0^{\frac{\pi}{4}} y(x)\,dx = \int_0^{\frac{\pi}{4}} 2\sin 2x + 3\cos 2x \,dx$

$= \left[-\cos 2x + \frac{3}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left[0 + \frac{3}{2}\right] - \left[-1 + 0\right]$

$= \frac{5}{2}$

c) $\text{VOLUME} = \pi\int_0^{\pi/4} \left(y(x)\right)^2\,dx = \pi\int_0^{\frac{\pi}{4}} \frac{13}{2} + \frac{5}{2}\cos 4x + 6\sin 4x \,dx$

$= \pi\left[\frac{13}{2}x + \frac{5}{8}\sin 4x - \frac{3}{2}\cos 4x\right]_0^{\frac{\pi}{4}}$

$= \pi\left\{\left[\frac{13\pi}{8} + 0 + \frac{3}{2}\right] - \left[0 + 0 - \frac{3}{2}\right]\right\}$

$= \pi\left[\frac{13}{8}\pi + 3\right]$

$= \frac{\pi}{8}\left[13\pi + 24\right]$

8. a)

IN : $\frac{dV}{dt} = 200$

OUT : $\frac{dV}{dt} = -kV$

NET : $\frac{dV}{dt} = 200 - kV$

C4, IYGB, PAPER Y

b) $\frac{dV}{dt} = 200 - kV$

$\Rightarrow \frac{1}{200-kV}\,dV = 1\,dt$

$\Rightarrow \int \frac{1}{200-kV}\,dV = \int 1\,dt$

$\Rightarrow \int \frac{-k}{200-kV}\,dV = \int -k\,dt$

$\Rightarrow \ln|200-kV| = -kt + C$

$\Rightarrow 200 - kV = e^{-kt+C}$

$\Rightarrow 200 - kV = Be^{-kt}$ $(B = e^{C})$

$\Rightarrow 200 - Be^{-kt} = kV$

$\Rightarrow \frac{200}{k} - \frac{B}{k}e^{-kt} = V$

$\Rightarrow V = \frac{200}{k} + Ae^{-kt}$ $\left(A = -\frac{B}{k}\right)$ //

c). $t=0 \quad V=0$ — 1

$t=10 \quad \frac{dV}{dt} = 100$ — 2

• BY (1) $0 = \frac{200}{k} + A$

$A = -\frac{200}{k}$

$\Rightarrow V = \frac{200}{k} - \frac{200}{k}e^{-kt}$

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$$\boxed{V = \frac{200}{k}\left(1 - e^{-kt}\right)}$$

$$\frac{dV}{dt} = \frac{200}{\cancel{k}}\left(\cancel{k}e^{-kt}\right)$$

$$\frac{dV}{dt} = 200e^{-kt}$$

- BY (2)

$$100 = 200e^{-10k}$$

$$\frac{1}{2} = e^{-10k}$$

$$e^{10k} = 2$$

$$10k = \ln 2$$

$$\boxed{k = \frac{1}{10}\ln 2}$$

- $$V = \frac{200}{\frac{1}{10}\ln 2}\left[1 - e^{-\left(\frac{1}{10}\ln 2\right)t}\right]$$

$$V = \frac{2000}{\ln 2}\left[1 - \left(e^{\ln 2}\right)^{-\frac{1}{10}t}\right]$$

$$V = \frac{2000}{\ln 2}\left[1 - 2^{-\frac{1}{10}t}\right]$$

AS REQUIRED