C4, IYGB, PAPER W

1. $ax^2 + xy - 2y^2 + b = 0$

Diff w.r.t x

$2ax + 1 \times y + x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$

$2ax + y + x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$

- NORMAL GRADIENT

$2y + 3x = 11$

$2y = -3x + 11$

$y = -\frac{3}{2}x + \frac{11}{2}$

- HENCE GRADIENT OF TANGENT AT $(1,4)$ MUST BE $\frac{2}{3}$

$\Rightarrow 2a \times 1 + 4 + 1 \times \frac{2}{3} - 4 \times 4 \times \frac{2}{3} = 0$

$\Rightarrow 2a + 4 + \frac{2}{3} - \frac{32}{3} = 0$

$\Rightarrow 2a + 4 - 10 = 0$

$\Rightarrow 2a = 6$

$\Rightarrow a = 3$

- ALSO $P(1,4)$ LIES ON C

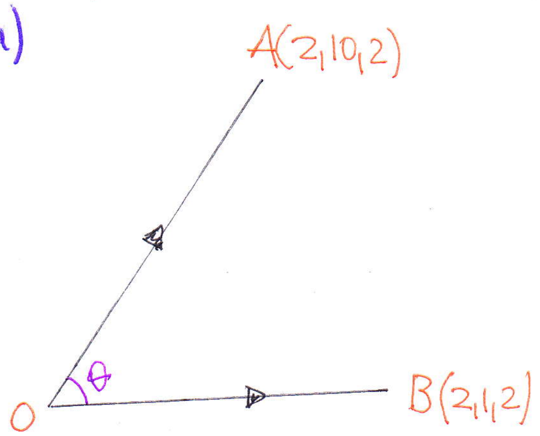
$ax^2 + xy - 2y^2 + b = 0$

$3 \times 1^2 + (1 \times 4) - 2 \times 4^2 + b = 0$

$3 + 4 - 32 + b = 0$

$b = 25$

2. a)



BY THE DOT PRODUCT

$\Rightarrow (2,10,2)\cdot(2,1,2) = |2,10,2||2,1,2|\cos\theta$

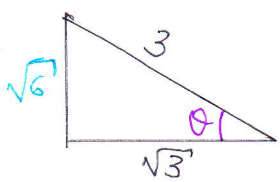
$\Rightarrow 4 + 10 + 4 = \sqrt{4+100+4}\sqrt{4+1+4}\cos\theta$

$\Rightarrow \cancel{18}^{\,6} = \sqrt{108} \times \cancel{3}\cos\theta$

$\Rightarrow 6 = 6\sqrt{3}\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

PYTHAGORAS $\sqrt{3^2 - \sqrt{3}^2}$



$\therefore \sin\theta = \frac{\sqrt{6}}{3}$

C4, IYGB, PAPER W

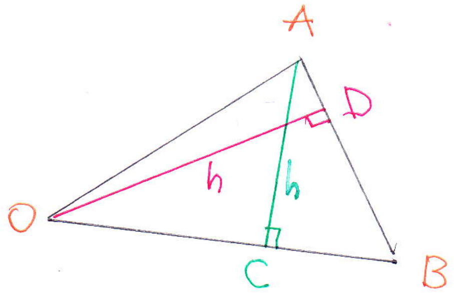
b) $\text{AREA} = \frac{1}{2}|OA||OB|\sin\theta$ (WHICH WE WORKED IN PART a)

$$= \frac{1}{2} \times 6\sqrt{3} \times \cancel{3} \times \frac{\sqrt{6}}{\cancel{3}}$$

$$= 3\sqrt{18}$$

$$= 9\sqrt{2}$$

c) I)



$$\text{AREA} = \frac{1}{2} \times |OB||AC|$$

$$9\sqrt{2} = \frac{1}{2} \times 3 \times |AC|$$

$$|AC| = 6\sqrt{2}$$

AS REQUIRED

II) $|\vec{AB}| = |\underline{b} - \underline{a}| = |(2,1,2) - (2,10,2)| = |0,-9,0| = 9$

$$\text{AREA} = \frac{1}{2}|AB||OD|$$

$$\cancel{9}\sqrt{2} = \frac{1}{2} \times \cancel{9} \times |OD|$$

$$|OD| = 2\sqrt{2}$$

AS REQUIRED

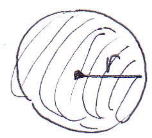
3. $\frac{dS}{dt} = 16$ GIVEN.

$$\frac{dV}{dt} = \frac{dV}{dS} \times \frac{dS}{dt}$$

$$\frac{dV}{dt} = \frac{dV}{dS} \times \frac{dS}{dt}$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dS} \times 16$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{8\pi r} \times 16$$



- $V = \frac{4}{3}\pi r^3$

  $\frac{dV}{dr} = 4\pi r^2$

- $S = 4\pi r^2$

  $\frac{dS}{dr} = 8\pi r$

- $S = 625\pi$

  $4\pi r^2 = 625\pi$

  $r = \frac{25}{2} = 12.5$

C4, IYGB, PAPER W

$$\frac{dV}{dt} = 8r$$

$$\left.\frac{dV}{dt}\right|_{S=625\pi} = \left.\frac{dV}{dt}\right|_{r=12.5} = 8 \times 12.5 = 100 \text{ cm}^3\text{s}^{-1}$$

4. $$\int \frac{e^{2x}-2e^{x}}{e^{x}+1}\,dx = \dots \text{ BY SUBSTITUTION}$$

$u = e^{x}+1$

$\frac{du}{dx} = e^{x}$

$dx = \frac{du}{e^{x}}$

$e^{x} = u-1$

$e^{x}-2 = u-3$

$$= \int \frac{e^{2x}-2e^{x}}{u}\,\frac{du}{e^{x}} = \int \frac{e^{2x}-2e^{x}}{ue^{x}}\,du$$

$$= \int \frac{e^{x}-2}{u}\,du = \int \frac{u-3}{u}\,du$$

$$= \int 1-\frac{3}{u}\,du = u-3\ln|u|+C$$

$$= e^{x}+1-3\ln(e^{x}+1)+C = e^{x}-3\ln(e^{x}+1)+C$$

5. a)

P = POPULATION (MILLIONS)

t = TIME (YEARS)

$t=0 \quad P=\frac{1}{4}$

$$\frac{dP}{dt} = +P^2(1-P)$$

↑ GROWING

$$\Rightarrow dP = P^2(1-P)\,dt$$

$$\Rightarrow \frac{1}{P^2(1-P)}\,dP = 1\,dt$$

$$\Rightarrow \int \frac{1}{P^2(1-P)}\,dP = \int 1\,dt$$

PARTIAL FRACTIONS

$$\frac{1}{P^2(1-P)} = \frac{A}{P^2} + \frac{B}{1-P} + \frac{C}{P}$$

$$1 \equiv A(1-P) + BP^2 + CP(1-P)$$

- IF $P=0 \Rightarrow \boxed{1 = A}$
- IF $P=1 \Rightarrow \boxed{1 = B}$
- IF $P=2 \Rightarrow 1 = -A + 4B - 2C$

$$1 = -1 + 4 - 2C$$
$$2C = 2$$
$$\boxed{C = 1}$$

$$\Rightarrow \int \frac{1}{P^2} + \frac{1}{1-P} + \frac{1}{P}\, dP = \int 1\, dt$$

$$\Rightarrow -\frac{1}{P} - \ln|1-P| + \ln P + C = t$$

APPLY CONDITION

$t=0,\ P=\frac{1}{4}$

$$\Rightarrow -4 - \ln\frac{3}{4} + \ln\frac{1}{4} + C = 0$$
$$\Rightarrow -4 + \ln\left(\frac{1/4}{3/4}\right) + C = 0$$
$$\Rightarrow C = 4 + \ln\frac{1}{3}$$
$$\Rightarrow \boxed{C = 4 - \ln 3}$$

$$-\frac{1}{P} - \ln|1-P| + \ln P + (4 - \ln 3) = t.$$

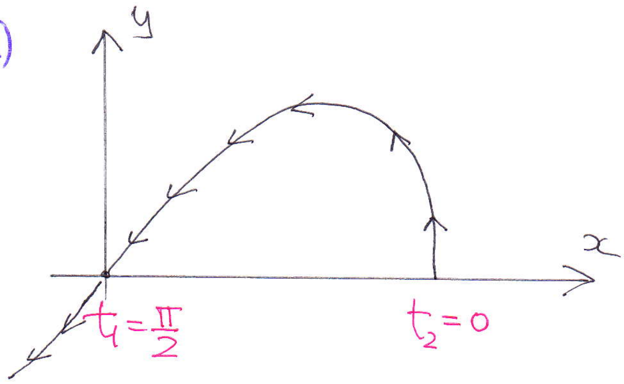
$$t = \ln\left|\frac{3P}{1-P}\right| - \frac{1}{P} + 4$$ AS REQUIRED

b) $\frac{dP}{dt} = P^2(1-P)$, $\frac{dP}{dt} = 0$ when $P = \cancel{0}$ / 1

∴ LIMITING VALUE IS 1

IE 1 MILLION

6. a)



$x = 6\cos t$

$y = 12\sin 2t$

$0 \leqslant t \leqslant 2\pi$

$y=0 \Rightarrow 12\sin 2t = 0$

$\Rightarrow \sin 2t = 0$

$\Rightarrow \begin{cases} 2t = 0 \pm 2n\pi \\ 2t = \pi \pm 2n\pi \end{cases}$ $n=0,1,2,3,\ldots$

$\Rightarrow \begin{cases} t = 0 \pm 4\pi \\ t = \pi/2 \pm 4\pi \end{cases}$

$x=-6$ $x=0$

$\therefore t = 0, \frac{\pi}{2}, \cancel{\pi}, \cancel{\frac{3\pi}{2}}$

$x=+6$ $x=0$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_{\frac{\pi}{2}}^{0} 12\sin 2t(-6\sin t)\,dt$$

$$= \int_{\frac{\pi}{2}}^{0} -72\sin 2t\sin t\,dt = \int_{\frac{\pi}{2}}^{0} -72(2\sin t\cos t)\sin t\,dt$$

$$= \int_{\pi/2}^{0} -144\sin^2 t\cos t\,dt = \int_{0}^{\frac{\pi}{2}} 144\cos t\sin^2 t\,dt$$

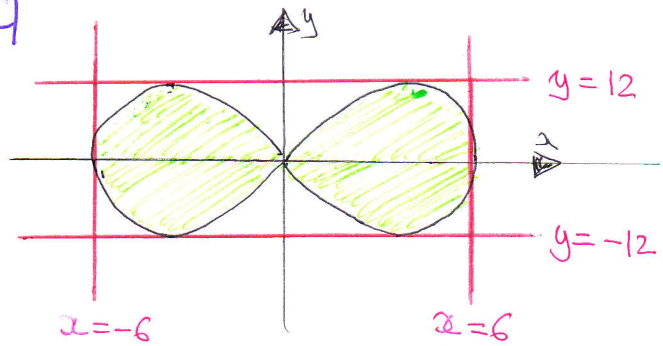
AS REQUIRED

b) INTEGRATE BY RECOGNITION

$$\ldots = \left[48\sin^3 t\right]_0^{\frac{\pi}{2}} = 48 - 0 = 48$$

$\therefore$ TOTAL AREA $= 4 \times 48 = 192$

d)



LOOKING AT

$x = 6\cos t \quad 0 \leq t \leq 2\pi$

$y = 12\sin 2t \quad 0 \leq t \leq 2\pi$

THUS THE AREA OF THE RECTANGLE IS $12 \times 24 = 288$

AREA OF THE CURVE (4 "LEAVES") $= 192$

$$\underline{\underline{96}}$$

7.

$$f(x) = \frac{1}{\sqrt{1-ax}} - \sqrt{1+bx}$$

$$= (1-ax)^{-\frac{1}{2}} - (1+bx)^{\frac{1}{2}}$$

$$= \left[1 + \frac{-\frac{1}{2}}{1}(-ax)^1 + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-ax)^2 + O(x^3)\right] - \left[1 + \frac{\frac{1}{2}}{1}(bx)^1 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(bx)^2 + O(x^3)\right]$$

$$= \left[1 + \frac{1}{2}ax + \frac{3}{8}a^2x^2 + O(x^3)\right] - \left[1 + \frac{1}{2}bx - \frac{1}{8}b^2x^2 + O(x^3)\right]$$

$$= \frac{1}{2}(a-b)x + \left(\frac{3}{8}a^2 - \frac{1}{8}b^2\right)x^2 + O(x^3)$$

$$= \underbrace{\frac{1}{2}(a-b)}_{2}x + \underbrace{\frac{1}{8}(3a^2+b^2)}_{26}x^2 + O(x^3)$$

HENCE

- $\frac{1}{2}(a-b) = 2$
  $a - b = 4$
  $\boxed{b = a-4}$

- $\frac{1}{8}(3a^2+b^2) = 26$
  $\Rightarrow 3a^2 + b^2 = 208$
  $\Rightarrow 3a^2 + (a-4)^2 = 208$
  $\Rightarrow 3a^2 + a^2 - 8a + 16 = 208$
  $\Rightarrow 4a^2 - 8a - 192 = 0$

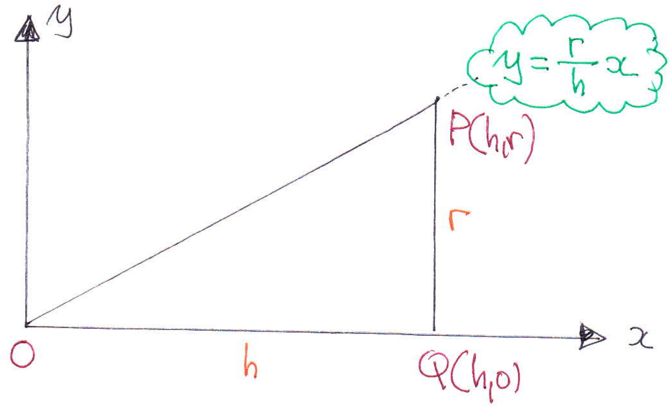
C4, IYGB, PAPER W

$\Rightarrow a^2 - 2a - 48 = 0$

$\Rightarrow (a-8)(a+6) = 0$

$\Rightarrow a = 8 \text{ or } \cancel{-6}$ & $b = 4 \text{ or } \cancel{10}$. $a > b > 0$

8.



$$V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx = \pi \int_0^h \left[\frac{r}{h}x\right]^2 dx = \pi \int_0^h \frac{r^2}{h^2}x^2 \, dx$$

$$= \pi \left[\frac{r^2}{3h^2}x^3\right]_0^h = \pi \left[\frac{r^2}{3\cancel{h^2}}\cancel{h^3} - 0\right] = \frac{1}{3}\pi r^2 h$$