Parametric equations

Standard topics/ various/parametric equations exam questions

Question 27 (***+)

A curve has parametric equations

$$x = t^2, \quad y = \frac{6}{t}, \quad t \in \mathbb{R}, \, t \neq 0 \, .$$

a) Determine a simplified expression for $\frac{dy}{dx}$, in terms of t.

b) Show that an equation of the tangent to the curve at the point $A(4,-3)$ is

$$3x - 8y - 36 = 0 \, .$$

c) Find the value of t at the point where the tangent to the curve at A meets the curve again.

$\boxed{\frac{dy}{dx} = -\frac{3}{t^3}}$, $\boxed{t = 4}$

Question 31 **(***+)**

A curve is given parametrically by the equations

$$x = 3t - 2\sin t, \quad y = t^2 + t\cos t \quad , \quad 0 \le t < 2\pi .$$

Show that an equation of the tangent at the point on the curve where $t = \frac{\pi}{2}$ is given by

$$y = \frac{\pi}{6}(x+2).$$

Question 34 (***+)

The curve C has parametric equations

$$x = \cos\theta,\ \ y = \sin 2\theta,\ \ 0 \le \theta < 2\pi .$$

The point P lies on C where $\theta = \dfrac{\pi}{6}$.

a) Find the gradient at P.

b) Hence show that the equation of the tangent at P is

$$2y + 4x = 3\sqrt{3} .$$

c) Show that a Cartesian equation of C is

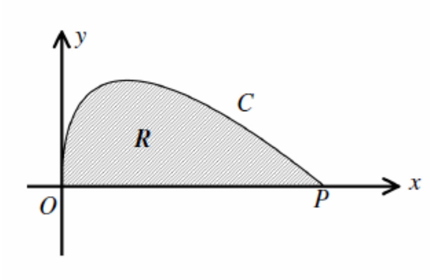
$$y^2 = 4x^2\left(1 - x^2\right).$$

$$\boxed{\left.\frac{dy}{dx}\right|_P = -2}$$

Parametric Integration

Standard topics/integration/ parametric integration exam questions

Question 5 (***)



The figure above shows the curve C, given parametrically by

$$x = 6t^2, \quad y = t - t^2, \quad t \geq 0.$$

The curve meets the x axis at the origin O and at the point P.

a) Show that the x coordinate of P is 6.

The finite region R, bounded by C and the x axis, is revolved in the x axis by 2π radians to form a solid of revolution, whose volume is denoted by V.

b) Show clearly that

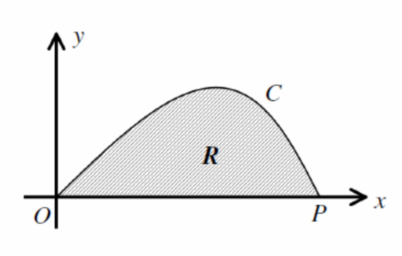
$$V = \pi \int_0^T 12t\left(t - t^2\right)^2 \, dt,$$

stating the value of T.

c) Hence find the value of V.

$\boxed{}$, $\boxed{T = 1}$, $\boxed{V = \frac{\pi}{5}}$

Question 7 (***)



The figure above shows the curve C, given parametrically by

$$x = 3t + \sin t, \quad y = 2\sin t, \quad 0 \le t \le \pi .$$

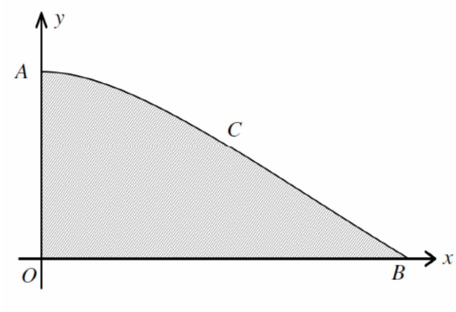
The curve meets the coordinate axes at the point P and at the origin O.

The finite region R is bounded by C and the x axis.

Determine the area of R.

$\boxed{}$, $\boxed{\text{area} = 12}$

Question 8 (***+)



The figure above shows the curve C, with parametric equations

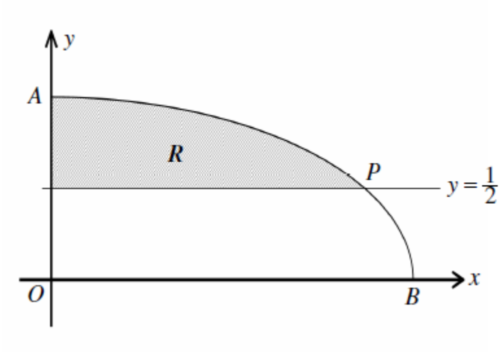
$$x = 36t^2 - \pi^2, \quad y = \frac{\sin 3t}{8}, \quad \frac{\pi}{6} \le t \le \frac{\pi}{3}.$$

The curve meets the coordinate axes at the points A and B.

By setting up and evaluating a suitable integral in parametric, show that the area bounded by C and the coordinate axes is $(\pi - 1)$ square units.

[ ], [proof]

Question 16 (****)



The figure above shows the curve C, with parametric equations

$$x = 4\cos\theta, \quad y = \sin\theta \ , \ 0 \le \theta \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B. The straight line with equation $y = \frac{1}{2}$ meets C at the point P.

a) Show that the area under the arc of the curve between A and P, and the x axis, is given by the integral

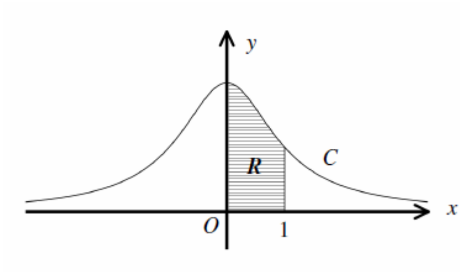
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \ d\theta .$$

The shaded region R is bounded by C, the straight line with equation $y = \frac{1}{2}$ and the y axis.

b) Find an exact value for the area of R.

$$\boxed{\text{area} = \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)}$$

Question 19 (****)



The figure above shows the curve C, defined by the parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

The finite region R is bounded by C, the coordinate axes and the straight line with equation $x = 1$.

a) Determine the area of R.

The region R is revolved by 2π radians in the x axis, forming a solid S.

b) Show that the volume of S is

$$\frac{\pi}{8}(\pi + 2).$$

c) Find a Cartesian equation of C, giving the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{\text{area} = \frac{\pi}{4}}$, $\boxed{y = \frac{1}{1+x^2}}$

Volumes of Revolution

Standard topics/integration/integration volume of revolution

Question 11 (***)

The curve C has equation

$$y = \sqrt{x} + \frac{4}{\sqrt{x}}, \quad x > 0.$$

The region bounded by C, the x axis and the lines $x = 1$, $x = 4$ is rotated through $360°$ about the x axis.

Show that the volume of the solid formed is

$$\frac{\pi}{2}(63 + 64\ln 2).$$

Question 25 **(***+)**

The curve C lies entirely above the x axis and has equation

$$y = 1 + \frac{1}{2\sqrt{x}}, \quad x \geq 0 .$$

a) Show that

$$y^2 = 1 + \frac{1}{\sqrt{x}} + \frac{1}{4x} .$$

The region R is bounded by the curve, the x axis and the straight lines with equations $x = 1$ and $x = 4$.

b) Show that when R is rotated by $360°$ about the x axis, the solid generated has a volume

$$\pi\left(5 + \ln\sqrt{2}\right) .$$

Differential Equations

Standard topics/odes separable no context

Question 3 (+)**

Find a general solution of the differential equation

$$\frac{dy}{dx} = (y+1)(1-2x), \ \ y \neq -1.$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A e^{x-x^2} - 1}$$

Question 4 **(**+)**

Find a general solution of the differential equation

$$\frac{dy}{dx} = y \tan x \, , \; y > 0$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = A \sec x}$$

Question 7 (*+)**

Find a general solution of the differential equation

$$\left(x^2+3\right)\frac{dy}{dx}=xy,\ \ y>0,$$

giving the answer in the form $y^2=f(x)$.

$$\boxed{y^2=A\left(x^2+3\right)}$$

Question 9 (*+)**

Find a general solution of the differential equation

$$\frac{dy}{dx} = \frac{x\mathrm{e}^x}{\sin y \cos y},$$

giving the answer in the form $f(x, y) = \text{constant}$.

$$\boxed{\cos 2y + 4\mathrm{e}^x (x-1) = C \quad \text{or} \quad \mathrm{e}^x (x-1) - \sin^2 y = C \quad \text{or} \quad \mathrm{e}^x (x-1) + \cos^2 y = C}$$

Question 6 (***)

The mass, m grams, of a burning candle, t hours after it was lit up, satisfies the differential equation

$$\frac{dm}{dt} = -k(m-10),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$m = 10 + Ae^{-kt},$$

where A is a non-zero constant.

The initial mass of the candle was 120 grams, and 3 hours later its mass has halved.

b) Find the value of A and show further that

$$k = \tfrac{1}{3}\ln\left(\tfrac{11}{5}\right).$$

c) Calculate, correct to three significant figures, the mass of the candle after a further period of 3 hours has elapsed.

$\boxed{}$, $\boxed{A = 110}$, $\boxed{m \approx 32.7}$

Question 8 (***)

The number of fish x in a small lake at time t months after a certain instant, is modelled by the differential equation

$$\frac{dx}{dt} = x(1-kt),$$

where k is a positive constant.

We may assume that x can be treated as a continuous variable.

It is estimated that there are 10000 fish in the lake when $t=0$ and 12 months later the number of fish returns back to 10000.

a) Find a solution of the differential equation, in the form $x = f(t)$.

b) Find the long term prospects for this population of fish.

$\boxed{x = 10000\mathrm{e}^{t-\frac{1}{12}t^2}}$, $\boxed{x \to 0}$

Question 10 (***+)

A cylindrical tank of height 150 cm is full of oil which started leaking out from a small hole at the side of a tank.

Let h cm be the height of the oil still left in the tank, after leaking for t minutes, and assume the leaking can be modelled by the differential equation

$$\frac{dh}{dt} = -\frac{1}{4}(h-6)^{\frac{3}{2}}.$$

a) Solve the differential equation to show that …

i. … $t = \frac{8}{\sqrt{h-6}} - \frac{2}{3}$.

ii. … $\sqrt{h-6} = \frac{24}{3t+2}$.

b) State how high is the hole from the bottom of the tank and hence show further that it takes 200 seconds for the oil level to reach 4 cm above the level of the hole.

$\boxed{6\text{cm}}$

Question 11 (***+)

Water is leaking out of a hole at the side of a tank.

Let the height of the water in the tank is y cm at time t minutes.

The rate at which the height of the water in the tank is decreasing is modelled by the differential equation

$$\frac{dy}{dt} = -6(y-7)^{\frac{2}{3}}.$$

When $t = 0$, $y = 132$.

a) Find how long it takes for the water level to drop from 132 cm to 34 cm.

The tank is filled up with water again to a height of 132 cm and allowed to leak out in exactly the same fashion as the one described in part **(a)**.

b) Determine how long it takes for the water to stop leaking.

$\boxed{}$, $\boxed{t = 1}$, $\boxed{t = 2.5}$

Trapezium rule

Standard topics/integration/Numerical integration

Question 3 (**)

The values of y, for the curve C with equation $y=\sqrt{x^3-x}$, have been tabulated below.

x	1	1.5	2	2.5	3	3.5	4
y	0	1.369	2.449	3.623			7.746

a) Complete the table.

b) Use the trapezium rule with all the values from the table to find an estimate, correct to 2 decimal places, for the integral

$$\int_1^4 \sqrt{x^3-x}\ dx.$$

$\boxed{}$, $\boxed{4.899}$, $\boxed{6.275}$, $\boxed{\approx 11.24}$