Algebraic fraction exam questions

Partial Fractions

Basic topics/various/algebraic fractions. P32-47

Question 1

Express each of the following into partial fractions.

a) $\dfrac{6x}{(x-1)(x+2)}$

b) $\dfrac{7y-11}{(y+2)(y-3)}$

$\boxed{\dfrac{2}{x-1}+\dfrac{4}{x+2}}$, $\boxed{\dfrac{2}{y-3}+\dfrac{5}{y+2}}$.

Question 7

Express each of the following into partial fractions.

a) $\dfrac{2x^2 - x - 3}{(x-2)(x-1)^2}$

b) $\dfrac{y^2 - 2y + 8}{(y+2)(y-2)^2}$

$\boxed{\frac{3}{x-2} - \frac{1}{x-1} + \frac{2}{(x-1)^2}}$, $\boxed{\frac{2}{(y-2)^2} + \frac{1}{y+2}}$,

Question 11

Express each of the following into partial fractions.

a) $\dfrac{2x^2-8x+5}{(x-1)(x-2)}$

b) $\dfrac{4x^2-5x-15}{(x+1)(x-2)}$

$\boxed{2+\frac{1}{x-1}-\frac{3}{x-2}}$, $\boxed{4+\frac{2}{x+1}-\frac{3}{x-2}}$,

Binomial expansion

Standard topics/various/binomial series expansion exam questions

Question 15 (***)

$$f(x) \equiv \frac{2-x}{\sqrt{1+x}}, \quad |x|<1.$$

a) Show that the first four terms in the binomial expansion of $f(x)$ are

$$2-2x+\frac{5}{4}x^2-x^3.$$

b) Use the answer of part (a) to find the first four terms in the expansion of

$$g(x)=\frac{2-2x}{\sqrt{1+2x}}.$$

$\boxed{g(x)=2-4x+5x^2-8x^3}$

Question 16 (***)

$$f(x) = \sqrt{1 + \frac{1}{8}x}, \quad |x| < 8.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^2.

b) By substituting $x = 1$ in the expansion, show clearly that

$$\sqrt{2} \approx \frac{256}{181}.$$

$$\boxed{f(x) = 1 + \frac{1}{16}x - \frac{1}{512}x^2 + O(x^3)}$$

Question 20 (***)

In the convergent binomial expansion of

$$(1+bx)^n, \quad |bx|<1$$

the coefficient of x is -6 and the coefficient of x^2 is 27.

a) Show that $b=3$ and find the value of n.

b) Find the coefficient of x^3.

c) State the range of values of x for which the above expansion is valid.

$\boxed{n=-2}$, $\boxed{[x^3]=-108}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$

Question 31 (***+)

In the series expansion of

$$(1+ax)^n,\ |ax|<1,\ \ a,n \in \mathbb{R},$$

the coefficient of x is 15 and the coefficients of x^2 and x^3 are equal.

a) Given that n is not a positive integer, show that $a=6$.

b) Find the value of n.

c) Find the coefficient of x^4.

$\boxed{}$, $\boxed{n=\frac{5}{2}}$, $\boxed{\left[x^4\right]=-\frac{405}{8}}$

Question 33 (***+)

The algebraic expression $\sqrt[3]{1-3x}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Find the first 4 terms in the series expansion of $\sqrt[3]{1-3x}$.

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the expansion, show that

$$\sqrt[3]{997} \approx 9.989989983.$$

$\boxed{1-x-x^2-\frac{5}{3}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{3}<x<\frac{1}{3}}$

Standard topics/various/implicit equations exam questions

Question 15 (***)

A curve has equation

$$2x^2 + xy + y^2 = 14.$$

a) Show clearly that

$$\frac{dy}{dx} = -\frac{4x+y}{x+2y}.$$

b) Hence, find the coordinates of the stationary points of the curve.

$\boxed{(1,-4),\ (-1,4)}$

Question 18 (***)

A curve C has implicit equation

$$x^2 - 4xy + y^2 = 13.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{x-2y}{2x-y}.$$

The points A and B are the two points on C whose x coordinate is 2.

b) Find the y coordinates of A and B.

The tangents to C at A and B, meet at the point P.

c) Find the exact coordinates of P.

$\boxed{}$, $\boxed{A(2,9),\ B(2,-1)}$, $\boxed{P\left(-\frac{13}{6}, -\frac{13}{3}\right)}$

Question 32 (***+)

A curve C has implicit equation

$$\sin 3x + \sin 2y = \sqrt{2}, \quad 0 \le x, y \le \frac{\pi}{3}.$$

The point P lies on C and its x coordinate is $\frac{\pi}{12}$.

a) Find the y coordinate of P.

b) Show that the gradient at P is $-\frac{3}{2}$.

c) Show further that the equation of the tangent to C at P is

$$4y + 6x = \pi.$$

$$\boxed{P\left(\frac{\pi}{12}, \frac{\pi}{8}\right)}$$

Question 36 (***+)

The equation of a curve is given by

$$e^y = \frac{x^2+3}{x-1}.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{(x-3)(x+1)}{(x^2+3)(x-1)}.$$

b) Find the exact coordinates of the turning point of the curve.

$\boxed{}$, $\boxed{(3, \ln 6)}$

Rates of Change

Standard topics/various/related rates of change

Question 5 (**+)

The volume, V cm^3, of a metallic cube of side length x cm, is increasing at the constant rate of 0.108 cm^{3}s^{-1}.

a) Determine the rate at which the side of the cube is increasing when the side length reaches 3 cm.

b) Find the rate at which the surface area of the cube, A cm^2, is increasing when the side length reaches 3 cm.

$\boxed{}$, $\boxed{\frac{1}{250} = 0.004 \text{ cm s}^{-1}}$, $\boxed{\frac{18}{125} = 0.144 \text{ cm}^2\text{ s}^{-1}}$

Question 8 (***)

Fine sand is dropping on a horizontal floor at the constant rate of $4 \text{ cm}^3\text{s}^{-1}$ and forms a pile whose volume, $V \text{ cm}^3$, and height, h cm, are connected by the formula

$$V = -8 + \sqrt{h^4 + 64}.$$

Find the rate at which the height of the pile is increasing, when the height of the pile has reached 2 cm.

$\boxed{}$, $\boxed{\sqrt{5} \approx 2.24 \text{ cm s}^{-1}}$

Question 14 (****)

Liquid is pouring into a container at the constant rate of $30 \text{ cm}^3\text{s}^{-1}$.

The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

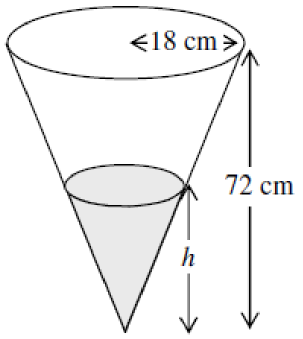
$$V = 36h^2 .$$

a) Find the rate at which the height of the liquid in the container is rising when the height of the liquid reaches 3 cm.

b) Determine the rate at which the height of the liquid in the container is rising 12.5 minutes after the liquid started pouring in.

$\boxed{}$, $\boxed{\frac{5}{36} = 0.139 \text{ cm s}^{-1}}$, $\boxed{\frac{1}{60} = 0.0167 \text{ cm s}^{-1}}$

Question 25 (****)



Flowers at a florists' are stored in vases which are in the shape of hollow inverted right circular cones with height 72 cm and radius 18 cm.

One such vase is initially empty and placed under a tap where the water is flowing into the vase at the constant rate of 6π cm^3 s^{-1}.

a) Show that the volume, V cm^3, of the water in the vase is given by

$$V = \frac{1}{48}\pi h^3,$$

where, h cm, is the height of the water in the vase.

b) Find the rate at which h is rising when $h = 4$ cm.

c) Determine the rate at which h is rising 12.5 **minutes** after the vase was placed under the tap.

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \frac{1}{3}\pi r^2 h\right]$

[ ], [6 cm s^{-1}], [$\frac{2}{75} \approx 0.0267$ cm^3 s^{-1}]

Vectors

Further topics/linear algebra/ vector exam questions part a

Question 2 (**)

Relative to a fixed origin O, the respective position vectors of three points A, B and C are

$$\begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ 11 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}.$$

a) Determine, in component form, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence find, to the nearest degree, the angle BAC.

c) Calculate the area of the triangle BAC.

$\boxed{\overrightarrow{AB} = -8\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}}$, $\boxed{\overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} - 17\mathbf{k}}$, $\boxed{\theta \approx 83°}$, $\boxed{\text{area} \approx 106}$

Question 3 (**)

The straight line l_1 passes through the points $A(2,5,9)$ and $B(6,0,10)$.

a) Find a vector equation for l_1.

The straight line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ 8 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix},$$

where μ is a scalar parameter.

b) Show that the point A is the intersection of l_1 and l_2.

c) Show further that l_1 and l_2 are perpendicular to each other.

$$\boxed{\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 9\mathbf{k} + \lambda(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})}$$

Question 4 (**)

Relative to a fixed origin O, the points A and B have respective position vectors

$$\mathbf{i}+7\mathbf{j}+5\mathbf{k} \text{ and } 5\mathbf{i}+\mathbf{j}-5\mathbf{k}.$$

a) Find a vector equation of the straight line l_1 which passes through A and B.

The straight line l_2 has vector equation

$$\mathbf{r}_2 = 5\mathbf{i}-4\mathbf{j}+4\mathbf{k}+\mu(\mathbf{i}-4\mathbf{j}+2\mathbf{k}),$$

where μ is a scalar parameter.

The point C is the point of intersection between l_1 and l_2.

b) Find the position vector of C.

c) Show that C is the midpoint of AB.

$$\boxed{\mathbf{r}=\mathbf{i}+7\mathbf{j}+5\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}-5\mathbf{k})}, \quad \boxed{\overrightarrow{OC}=3\mathbf{i}+4\mathbf{j}}$$

Question 6 (**+)

The points $A(2,4,4)$, $B(6,8,4)$, $C(6,4,0)$, $D(2,0,0)$ and $M(4,4,2)$ are given.

The straight line l_1 has equation

$$\mathbf{r}_1 = 6\mathbf{i} + 4\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}),$$

where λ is a scalar parameter.

The straight line l_2 passes through the points C and M.

a) Find a vector equation of l_2.

b) Show that $\overrightarrow{AB}$ is parallel to l_1.

c) Verify that D lies on l_1.

d) Find the acute angle between $\overrightarrow{AC}$ and l_1.

$\boxed{\mathbf{r}_2 = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{k})}$, $\boxed{60°}$

Question 9 (**+)

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}_2 = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k}),$$

where λ and μ are scalar parameters.

a) Show that l_1 and l_2 **do not** intersect.

The point P lies on l_1 where $\lambda = 4$ and the point Q lies on l_2 where $\mu = -1$.

b) Find the acute angle between PQ and l_1.

$\boxed{}$, $\boxed{56.3°}$

Question 19 (***)

With respect to a fixed origin O, the point A has position vector $8\mathbf{i}-6\mathbf{j}+5\mathbf{k}$ and the point B has position vector $t\mathbf{i}+t\mathbf{j}+2t\mathbf{k}$.

a) Show clearly that

$$|AB|^2 = 6t^2 - 24t + 125.$$

Let $f(t) = 6t^2 - 24t + 125$.

b) Find the value of t for which $f(t)$ takes a minimum value.

c) Hence determine the closest distance between A and B.

$\boxed{t=2}$, $\boxed{\sqrt{101}}$

Question 20 (***)

Relative to a fixed origin O, the points A and B have respective coordinates

$$(2,-3,3) \quad \text{and} \quad (5,1,b),$$

where b is a constant.

The point C is such so that $OABC$ is a rectangle, where O is the origin.

a) Show clearly that $b=5$.

b) Determine the position vector of C.

c) Find the vector equation of the straight line l that passes through A and C.

$$\boxed{C(3,4,2)}, \ \boxed{\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 7\mathbf{j} - \mathbf{k})}$$

Question 24 (***)

The straight line L_1 passes through the points $A(3,0,3)$ and $B(5,5,2)$.

The straight line L_2 has a vector equation given by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$$

where μ is a scalar parameter.

a) Write down the coordinates of the point of intersection of L_1 and L_2.

b) Find the size of the acute angle θ, between L_1 and L_2.

c) Calculate the distance AB.

The point C lies on L_1 so that the distance AB is equal to the distance AC.

d) Determine the coordinates of C.

$\boxed{P(5,5,2)}$, $\boxed{\theta \approx 73.2°}$, $\boxed{|AB| = \sqrt{30}}$, $\boxed{C(1,-5,4)}$

Question 35 (***)

The points with coordinates $A(8,0,12)$ and $B(9,-2,14)$ are given.

a) Find the vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 9\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 are perpendicular.

c) Show further that l_1 and l_2 intersect at some point P and state the coordinates of P.

The point $C(9,13,2)$ lies on l_2 and the point D is the reflection of C about l_1.

d) Determine the coordinates of D.

$\boxed{}$, $\boxed{\mathbf{r}_1 = 8\mathbf{i} + 12\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})}$, $\boxed{P(3,10,2)}$, $\boxed{D(-3,7,2)}$

Question 36 (***)

Relative to a fixed origin O, the point A has position vector $7\mathbf{i}+4\mathbf{j}$ and the point B has position vector $-3\mathbf{j}+7\mathbf{k}$. The straight line L_1 passes through the points A and B.

a) Find a vector equation for L_1.

The straight line L_2 has a vector equation

$$\mathbf{r} = 3\mathbf{i}-2\mathbf{j}-4\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+3\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that L_1 and L_2 intersect at some point C, and find its position vector.

c) Show further that L_1 and L_2 are perpendicular.

The point D has position vector $4\mathbf{i}-\mathbf{k}$.

d) Verify that D lies on L_2.

The point E is the image of D after reflection about L_1.

e) Find the position vector of E.

$\boxed{}$, $\boxed{\mathbf{r}=7\mathbf{i}+4\mathbf{j}+\lambda(\mathbf{i}+\mathbf{j}-\mathbf{k})}$, $\boxed{\overrightarrow{OC}=5\mathbf{i}+2\mathbf{j}+2\mathbf{k}}$, $\boxed{\overrightarrow{OE}=6\mathbf{i}+4\mathbf{j}+5\mathbf{k}}$

Question 52 (***+)

The straight lines L_1 and L_2 have vector equations

$$\mathbf{r}_1 = \begin{pmatrix} 12 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 21 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},$$

where λ and μ are scalar parameters.

a) Show that L_1 and L_2 intersect at the point P, and find its coordinates .

b) Show further that L_1 and L_2 are perpendicular to each other.

The point $A(0,1,21)$ lies on L_2 and the point B lies on L_1 so that $|\overrightarrow{AP}| = |\overrightarrow{PB}|$.

c) Find the distance AB.

d) Hence state the shortest distance of P from the line through A and B.

$\boxed{P(6,13,3)}$, $\boxed{|AB| = 12\sqrt{7}}$, $\boxed{6\sqrt{7}}$

Question 67 (****)

The points A and B have position vectors $9\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ and $9\mathbf{i}+4\mathbf{j}+\mathbf{k}$, respectively.

a) Find a vector equation of the straight line l_1 that passes through A and B.

The straight line l_2 has the vector equation

$$\mathbf{r}_2 = 6\mathbf{i}+3\mathbf{j}-4\mathbf{k}+\mu(\mathbf{i}+\mathbf{j}-\mathbf{k}),$$

where μ is a scalar parameter.

b) Show that l_1 and l_2 intersect and find the position vector of their point of intersection.

c) Find the acute angle between l_1 and l_2.

The point C lies on l_2 in such a position so that is closest to A.

d) Show that the position vector of C is given by

$$\mathbf{c} = 4\mathbf{i}+\mathbf{j}-2\mathbf{k}.$$

$\boxed{\mathbf{r}_1 = 9\mathbf{i}+3\mathbf{j}+5\mathbf{k}+\lambda(\mathbf{j}-4\mathbf{k})}$, $\boxed{9\mathbf{i}+6\mathbf{j}-7\mathbf{k}}$, $\boxed{45.6°}$

Integration

Standard topics/integration/integration structured exam questions

Question 20 (**+)

By using the substitution $u = \ln x$, or otherwise, find an exact value for

$$\int_{e}^{3} \frac{1}{x \ln x} \, dx .$$

$\boxed{\ln(\ln 3)}$

Question 21 (**+)

$$f(x) \equiv \frac{x-5}{x^2+5x+4}.$$

a) Express $f(x)$ in partial fractions.

b) Find the value of

$$\int_0^2 f(x)\ dx\ ,$$

giving the answer as a single simplified logarithm.

$\boxed{}$, $\boxed{f(x) \equiv \frac{3}{x+4} - \frac{2}{x+1}}$, $\boxed{\int_0^2 f(x)\ dx = \ln\left(\frac{3}{8}\right)}$

Question 23 (+)**

Use the substitution $u = \sqrt{2x-7}$ to find

$$\int_4^8 \frac{6x}{\sqrt{2x-7}}\, dx .$$

$\boxed{}$, $\boxed{68}$

Question 27 (***)

$$\frac{x^2+3}{x-1} \equiv Ax + B + \frac{C}{x-1}.$$

a) Determine the value of each of the constants A, B and C.

b) Hence, or otherwise, evaluate

$$\int_2^4 \frac{x^2+3}{x-1}\, dx,$$

giving the answer in terms of natural logarithms.

$\boxed{A=1}$, $\boxed{B=1}$, $\boxed{C=4}$, $\boxed{8+4\ln 3}$

Question 29 (***)

By using the substitution $u = 3x + 1$, or otherwise, find

$$\int_0^5 x\sqrt{3x+1}\ dx .$$

$$\boxed{\frac{204}{5} = 40.8}$$

Question 30 (***)

Use an appropriate integration method to find an exact value for

$$\int_{0}^{\frac{\pi}{3}} 6x \sin 3x \; dx .$$

$\boxed{}$, $\boxed{\frac{2\pi}{3}}$

Question 31 (***)

By using the substitution $u = \sec x$, or otherwise, find

$$\int \tan x \sec^4 x \, dx .$$

$$\boxed{\frac{1}{4}\sec^4 x + C}$$

Question 35 (***)

Find an expression for the integral

$$\int \frac{3x-10}{x^2+5x-6}\,dx.$$

$$\boxed{4\ln|x+6|-\ln|x-1|+C}$$

Question 37 (***)

Use integration by parts to find the value of

$$\int_1^{e} \ln x \ dx .$$

$\boxed{}$, $\boxed{1}$

Question 41 (***)

$$\frac{8x-1}{(2x-1)^2} \equiv \frac{A}{2x-1} + \frac{B}{(2x-1)^2}.$$

a) Determine the value of each of the constants A and B.

b) Hence find the exact value of

$$\int_1^{1.5} \frac{8x-1}{(2x-1)^2}\, dx.$$

$\boxed{A=4}$, $\boxed{B=3}$, $\boxed{\frac{3}{4}+2\ln 2}$

Question 45 (***)

Use the substitution $u = 10\cos x - 1$ to find

$$\int_0^{\frac{\pi}{3}} 15(10\cos x - 1)^{\frac{1}{2}} \sin x \ dx .$$

$\boxed{19}$

Question 46 (***)

$$\frac{2x^2-x+6}{x^2(3-2x)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{3-2x}.$$

a) Determine the value of each of the constants A, B and C.

b) Evaluate

$$\int_2^3 \frac{2x^2-x+6}{x^2(3-2x)}\, dx,$$

giving the answer in the form $p-\ln q$, where p and q are constants.

$\boxed{A=1}$, $\boxed{B=2}$, $\boxed{C=4}$, $\boxed{\frac{1}{3}-\ln 6}$

Question 51 **(***+)**

Use trigonometric identities to find

$$\int \sec^2 x\left(1+\cot^2 x\right)\,dx\,.$$

$\boxed{\tan x - \cot x + C}$

Question 77 (***+)

It is given that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

a) Prove the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x+x)$.

b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \; dx.$$

$\boxed{\frac{2}{3}}$

Question 89 (***+)

Use trigonometric identities to integrate

$$\int \frac{\cos 2x}{1-\cos^2 2x}\,dx.$$

$\boxed{}$, $\boxed{-\frac{1}{2}\operatorname{cosec} 2x + C}$