A1 DOUBLES ASSIGNMENT 23 – PART A

Drill

1)

Prove these identities.

a $1 - \cos 2A \equiv \tan A \sin 2A$

b $\sin 2A(1 + \tan^2 A) \equiv 2\tan A$

c $\cot A - \tan A \equiv 2\cot 2A$

Ans. proof

2)

Work out the binomial expansions of these expressions up to and including the term in x^3

a $(1+x)^{-3}$ **b** $(1+x)^{\frac{1}{2}}$

c $(1+x)^{\frac{2}{3}}$ **d** $(1+4x)^{-1}$

e $(1-3x)^{-2}$ **f** $\left(1+\frac{1}{2}x\right)^{\frac{1}{3}}$

1 **a** $1 - 3x + 6x^2 - 10x^3 + \ldots$ **b** $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$

c $1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\ldots$ **d** $1-4x+16x^2-64x^3+\ldots$

e $1+6x+27x^2+108x^3+\ldots$ **f** $1+\frac{1}{6}x-\frac{1}{36}x^2+\frac{5}{648}x^3+\ldots$

PROBLEM SOLVING

1.

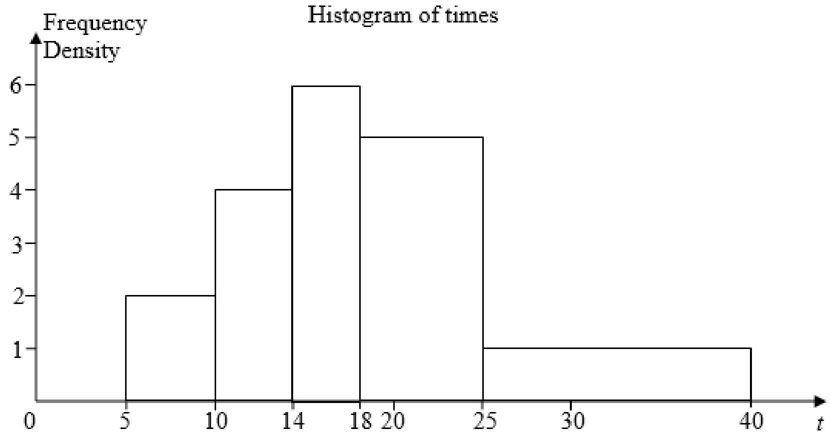

Figure 2

Figure 2 shows a histogram for the variable *t* which represents the time taken, in minutes, by a group of people to swim 500 m.

(*a*) Copy and complete the frequency table for *t*.

t	5 – 10	10 – 14	14 – 18	18 – 25	25 – 40
Frequency	10	16	24		

(*b*) Estimate the number of people who took longer than 20 minutes to swim 500 m.

(*c*) Find an estimate of the mean time taken.

(*d*) Find an estimate for the standard deviation of *t*.

(*e*) Find the median and quartiles for *t*.

One measure of skewness is found using $\frac{3(\text{mean} - \text{median})}{\text{standard deviation}}$.

(*f*) Evaluate this measure and describe the skewness of these data.

ANS: (a) 35, 15 (b) 40 (c) 18.91 (d) 7.26 (e) 13.75, 18, 23 (f) 0.376… positive skew

2. Normal distribution basics

A packing plant fills bags with cement. The weight X kg of a bag of cement can be modelled by a normal distribution with mean 50 kg and standard deviation 2 kg.

(a) Find $P(X > 53)$.

(b) Find the weight that is exceeded by 99% of the bags.

Three bags are selected at random.

(c) Find the probability that two weigh more than 53 kg and one weighs less than 53 kg.

ANS: (a) 1-0.9332 (b) $x_0 = 45.3474$ (c) $= 0.012492487..$

3. Normal distribution continuity correction binomial approx

For a particular type of plant 45% have white flowers and the remainder have coloured flowers. Gardenmania sells plants in batches of 12. A batch is selected at random.

Calculate the probability this batch contains

(a) exactly 5 plants with white flowers,

(b) more plants with white flowers than coloured ones.

Gardenmania takes a random sample of 10 batched of plants.

(c) Find the probability that exactly 3 of these batches contain more plants with white flowers than coloured ones.

Due to an increasing demand for these plants by large companies, Gardenmania decides to sell them in batches of 50.

(d) Use a suitable approximation to calculate the probability that a batch of 50 plants contains more than 25 plants with white flowers.

ANS: (a) 0.2225 (b) 0.2607 (c) 0.2567 (d) 0.1977

4. Compound/double angle solves

(a) Starting from the formulae for $\sin(A + B)$ and $\cos(A + B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

(*b*) Deduce that

$$\tan\left(\theta + \frac{\pi}{6}\right) = \frac{1 + \sqrt{3}\tan\theta}{\sqrt{3} - \tan\theta}.$$

(*c*) Hence, or otherwise, solve, for $0 \le \theta \le \pi$,

$$1 + \sqrt{3}\tan\theta = (\sqrt{3} - \tan\theta)\tan(\pi - \theta).$$

Give your answers as multiples of π.

ANS: $\theta = \frac{5}{12}\pi$, $\frac{11}{12}\pi$

5. Trapezium Rule

a) Use the trapezium rule with 4 equally spaced strips to find an estimate for

$$\int_0^1 2^{\sqrt{x}}\, dx.$$

b) Use the answer of part (a) to find estimates for

$$\int_0^1 2^{\sqrt{x}} + 3\, dx$$

$$\int_0^1 2^{\sqrt{x}+3}\, dx$$

ANS: a) 3.901, b)9.901, c)31.21

6. Exponential function and log rules & modelling logarithmic relationships

The data below follows a trend of the form $y = ab^x$, where a and b are constants

x	2	3	5	6.5	9
y	124.8	424.4	4097.0	30763.6	655743.5

(a) Copy and complete the table of values x and log y, giving your answers to 2 d.p.

x	2	3	5	6.5	9
log y	2.10				

(b) Plot a graph of log y against x and draw the line of best fit.

(c) Use your graph to estimate the values of a and b to one decimal place.

ANS: (a) missing values 2.63, 3.61, 4.49, 5.82
(c) b = 3.4, a = 10

7. Proof (exhaustion, counter-example, deduction, geometric problems)

The digits of a 2-digit number are different. A new number is formed by reversing the digits. Prove that the difference between the numbers is a multiple of 9.

ANS:

8. Small angle Maclauren Approx
Using your knowledge of the small angle approximations for $sinx$ and $cosx$, and the compound angle formulae, show from first principles, that

$$\frac{d(cosx)}{dx} = -sinx$$

ANS:
Google the proof if you need help!

9. Find algebraically the exact solutions to the equations

(*a*) $\ln(4 - 2x) + \ln(9 - 3x) = 2\ln(x + 1), \quad -1 < x < 2,$

(*b*) $2^x e^{3x+1} = 10.$

Give your answer to (*b*) in the form $\frac{a + \ln b}{c + \ln d}$ where *a*, *b*, *c* and *d* are integers.

.

ANS: (a) $x = \frac{7}{5}$ (b) $x = \frac{-1 + \ln 10}{3 + \ln 2}$

10. (*a*) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(*b*) $$p(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of $p(\theta)$,

(ii) the value of θ at which the maximum occurs.

ANS:

(a) 10; 0.927 (b) (i) $p(x) = \dfrac{4}{12-10}$ max 2 (ii) 4.07

11. Given that

$$2\cos(x+50)° = \sin(x+40)°.$$

(*a*) Show, without using a calculator, that

$$\tan x° = \frac{1}{3}\tan 40°.$$

(*b*) Hence solve, for $0 \le \theta < 360$,

$$2\cos(2\theta+50)° = \sin(2\theta+40)°,$$

giving your answers to 1 decimal place.

ANS: $\theta = 7.8$, 97.8, 187.8, 277.8

12. (*a*) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \qquad |x| < 1$$

(*b*) Substitute $x = \dfrac{1}{26}$ into

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = 1 + x + \frac{1}{2}x^2$$

to obtain an approximation to $\sqrt{3}$.

Give your answer in the form $\dfrac{a}{b}$ where a and b are integers.

ANS: $\sqrt{3} = \dfrac{7025}{4056}$

A1 DOUBLES ASSIGNMENT 23 – PART B

Drill

1 For each expression, find $\frac{dy}{dx}$ in terms of x and y

a $x^2 = y^3$

b $(x+1)^2 + (y-3)^2 = 4$

c $2x^2 + y^2 = 4$

d $\frac{1}{x} + \frac{1}{y} = 1$

e $x^2 + 2xy + 3y = 0$

f $y + \frac{1}{y} = x^2$

Ans.

1 a $\frac{2x}{3y^2}$ b $-\frac{x+1}{y-3}$

c $-\frac{2x}{y}$ d $-\frac{y^2}{x^2}$

e $-\frac{2(x+y)}{2x+3}$ f $\frac{2xy^2}{y^2-1}$

2 Work out the Cartesian equations given by these parametric equations.

a $x = t^2 + 2;\ y = 2t + 1$ b $x = 2t;\ y = -4t^3$

c $x = 3t;\ y = \frac{1}{2t}$ d $x = \frac{-2}{t};\ y = 3t^4$

e $x = \frac{1+t}{1-t};\ y = 2t$ f $x = 2\sin\theta;\ y = 2\cos\theta$

Ans.

2 **a** $(y-1)^2 = 4x - 8$ **b** $y = -\frac{x^3}{2}$

c $y = \frac{3}{2x}$ **d** $y = \frac{48}{x^4}$

e $y = \frac{2(x-1)}{1+x}$ **f** $x^2 + y^2 = 4$

PROBLEM SOLVING

1. A car is moving along a straight road with uniform acceleration. The car passes a check-point *A* with a speed of 12 m s^{-1} and another check-point *C* with a speed of 32 m s^{-1}. The distance between *A* and *C* is 1100 m.

(a) Find the time, in seconds, taken by the car to move from *A* to *C*.

Given that *B* is the mid-point of *AC*,

(b) find, in m s^{-1} to 1 decimal place, the speed with which the car passes *B*.

ANS: (a) 50 seconds (b) 24.2 ms^{-1}

2. A particle is moving on an axis *Ox*. From time $t = 0$ s to time $t = 32$ s, the particle is travelling with constant speed 15 m s^{-1}. The particle then decelerates from 15 m s^{-1} to rest in *T* seconds.

a) Sketch a speed-time graph to illustrate the motion of the particle.

The total distance travelled by the particle is 570m.

b) Find the value of *T*.
c) Sketch a distance-time graph illustrating the motion of the particle.

ANS: (b) 12 seconds

3. Forces and Newton's laws: single particles static tension

A stone that is modelled as a particle of mass *m* kg sits on a smooth slope of angle 30^o to the horizontal, it is held at rest by a horizontal force of PN. Given that the reaction force between the particle and the slope is 2N, find the mass and the value of P.

ANS: m = 0.18 kg P = 1 N

4. Differentiation - 1st Principles, gradient, normals, tangents, increasing/decreasing functions

Use first principles to show that the derivative of x^3 is $3x^2$ – **don't forget the final sentence**!

ANS: Proof

5. Implicit Differentiation

A curve C has equation $2\ cos3x\ siny = 1,\ x \geq 0, y \leq \pi$
The point $P(\frac{\pi}{12}, \frac{\pi}{4})$ lies on C.
Find an equation of the tangent to C at the point P.

ANS: y = 3x

6. A curve has parametric equations $x = t^3 - 12t, \quad y = 3t^2$
a) Find the equation of the normal at the point with parameter t = 1
b) Find the exact co-ordinates of the points where this normal cuts the curve again

ANS: 6) a) $y - 3 = \frac{3}{2}(x + 11)$ b) $A\left(14 + \sqrt{53}, \frac{81+3\sqrt{53}}{2}\right)$ $B\left(14 - \sqrt{53}, \frac{81-3\sqrt{53}}{2}\right)$

7. Integration as limit of a sum.

Evaluate the following

$$\lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{4}} \sin\left(2x + \frac{\pi}{4}\right) dx$$

ANS: $\frac{\sqrt{2}}{2}$

8. Integrate the following using standard integrals, recognition, partial fractions, substitution or parts:

(a) $\int \left(1 - \frac{1}{x}\right)^2 \mathrm{d}x$ (b) $\int \sin\frac{x}{2}\cos^4\frac{x}{2}\ \mathrm{d}x$ (c) $\int (\sin x + 2\cos x)^2\ \mathrm{d}x$

(d) $\int (4 - 5x)^{-1}\ \mathrm{d}x$ (e) $\int 3\ln x\,\mathrm{d}x$ (f) $\int \tan 3x\ \mathrm{d}x$

(g) $\int x^2 \sin x\ \mathrm{d}x$

(h) By substitution, show that $\int_0^3 15x\sqrt{x+1}\ dx = 116$ let u = x+1

(i) $\int \cot 5x\ \mathrm{d}x$ (j) $\int \frac{x+2}{(x-1)}\mathrm{d}x$ (k) $\int (2 - 3x)^{-2}\ \mathrm{d}x$

(l) $\int \operatorname{cosec}2x\cot 2x\,dx$ (m) $\int \frac{x^2}{x^2-4}dx$ (n) $\int \sec^2 x\, e^{\tan x} dx$

(o) $\int \frac{x}{9x^2+1}dx$ (p) $\int \sec 3x\, dx$

ANS: 8) a) $x-2\ln x-\frac{1}{x}+c$ b) $-\frac{2}{5}\cos^5\frac{x}{2}+c$ c) $\frac{5x}{2}+\frac{3}{4}\sin 2x+2\sin^2 x+c$

d) $-\frac{1}{5}\ln|4-5x|+c$ e) $3x\ln x-3x+c$ f) $\frac{1}{3}\ln|\sec 3x|+c$

g) $-x^2\cos x+2x\sin x+2\cos x+c$ h) proof i) $\frac{1}{5}\ln|\sin 5x|+c$

j) $x+3\ln|x-1|+c$ k) $\frac{1}{3}(2-3x)^{-1}+c$ l) $-\frac{1}{2}\operatorname{cosec}2x+c$

m) $-\frac{1}{2}\operatorname{cosec}2x+c$ n) $e^{\tan x}+c$ o) $\frac{1}{18}\left(\ln|9x^2+1|\right)+c$ p) $\frac{1}{3}\ln|\sec 3x+\tan 3x|+c$

9. Use the substitution $u = 1 - x^2$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx$$

ANS: 9) use your calculator to check

10. By using a suitable substitution, find the exact value of

$$\int_0^1 \frac{2^x}{2^x+1}dx$$

ANS: 10) use your calculator to check

11. Use integration by parts to find the exact value of

$$\int_1^3 x^2 ln x\, dx$$

ANS: 11) Use your calculator to check

12. Functions; Composites, domain, range, inverses

The function f is defined by

$$f: x \mapsto \frac{2(x-1)}{x^2-2x-3} - \frac{1}{x-3}, \quad x > 3.$$

(*a*) Show that $f(x) = \frac{1}{x+1}, \quad x > 3.$

(*b*) Find the range of f.

(*c*) Find $f^{-1}(x)$. State the domain of this inverse function.

The function g is defined by

$$g: x \mapsto 2x^2 - 3, \quad x \in \mathbb{R}.$$

(*d*) Solve $fg(x) = \frac{1}{8}$.

ANS:

(a) Proof

(b) $0 < f(x) < 1/4$

(c) $f^{-1}(x) = \frac{1}{x} - 1, \quad 0 < x < \frac{1}{4}$

(d) $x = \pm\sqrt{5}$