α	β	γ	δ	ε	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	π	ρ	σ	τ	υ	φ	χ	ψ	ω

"The mathematician's patterns, like the painter's or the poet's, must be beautiful: the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test." G H Hardy

Further Maths A2 (M2FP2D1) Assignment ρ (rho) A
Due 29th January 18

PREPARATION *Every week you will be required to do some preparation for future lessons, to be advised by your teacher.*

CURRENT WORK – MECHANICS Collisions

1. A small smooth sphere falls from rest onto a smooth horizontal plane. It takes 3 seconds to reach the plane. The coefficient of restitution between the particle and the plane is 0.49. Find the time it takes for the sphere to reach the plane a second time.

2. Three small smooth spheres A, B and C of equal radius have masses 500 g, 500 g and 1 kg respectively. The spheres move along the same straight line on a horizontal plane with A following B which is following C. Initially the velocities of A, B and C are $4\mathbf{i}\,\text{m s}^{-1}$, $-2\mathbf{i}\,\text{m s}^{-1}$ and $0.5\mathbf{i}\,\text{m s}^{-1}$ respectively, where $\mathbf{i}$ is a unit vector in the direction ABC. Sphere A collides with sphere B and then sphere B collides with sphere C. The coefficient of restitution between A and B is $\frac{2}{3}$ and between B and C is $\frac{1}{2}$. Find the velocities of the three spheres after all of the collisions have taken place.

3. Three perfectly elastic particles A, B and C of masses $3m$, $5m$ and $4m$ respectively lie at rest on a straight line on a smooth horizontal table with B between A and C. Particle A is projected directly towards B with speed $6\,\text{m s}^{-1}$ and after A has collided with B, B then collides with C. Find the speed of each particle after the second impact.

4. Three identical spheres A, B and C of equal mass m, and equal radius move along the same straight line on a horizontal plane. B is between A and C. A and B are moving towards each other with velocities $4u$ and $2u$ respectively while C moves away from B with velocity $3u$.

   a If the coefficient of restitution between any two of the spheres is e, show that B will only collide with C if $e > \frac{2}{3}$.

   b Find the direction of motion of A after collision, if $e > \frac{2}{3}$.

5. Two particles P of mass $2m$ and Q of mass $3m$ are moving towards each other with speeds $4u$ and $2u$ respectively. The direction of motion of Q is reversed by the impact and its speed after impact is u. This particle then hits a smooth vertical wall perpendicular to its direction of motion. The coefficient of restitution between Q and the wall is $\frac{2}{3}$. In the subsequent motion, there is a further collision between Q and P. Find the speeds of P and Q after this collision.

6. A bullet of mass 0.15 kg moving horizontally at 402 m s^{-1} embeds itself in a sandbag of mass 30 kg, which is suspended freely. Assuming that the sandbag is stationary before the impact, find

a the common velocity of the bullet and the sandbag,

b the loss of kinetic energy due to the impact.

7. A truck of mass 5 tonnes is moving at 1.5 m s^{-1} when it hits a second truck of mass 10 tonnes which is at rest. After the impact the second truck moves at 0.6 m s^{-1}. Find the speed of the first truck after the impact and the total loss of kinetic energy due to the impact.

8. A particle of mass m moves in a straight line with velocity v when it explodes into two parts, one of mass $\frac{1}{3}m$ and the other of mass $\frac{2}{3}m$ both moving in the same direction as before. If the explosion increases the energy of the system by $\frac{1}{4}mu^2$, where u is a positive constant, find the velocities of the particles immediately after the explosion. Give your answers in terms of u and v.

9. Two particles, A and B, of masses m and M respectively, are connected by a light inextensible string. The particles are side by side on a smooth floor and A is projected with speed u directly away from B. When the string becomes taut particle B is jerked into motion and A and B then move with a common speed in the direction of the original velocity of A. Find the common speed of the particles after the string becomes taut, and show that the loss of total kinetic energy due to the jerk is $\frac{mMu^2}{2(m+M)}$.

CONSOLIDATION – FP2

10. $$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}, \; t \geq 0$$

a) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found.

b) Find the general solution of the differential equation.

Given that a particular solution satisfies y = 3 and $\frac{dy}{dt} = 1$ when t = 0,

c) find this solution.

Another particular solution which satisfies y = 1 and $\frac{dy}{dt} = 0$ when t = 0, has equation

$$y = \left(1 - 3t + 2t^2\right)e^{3t}$$

d) For this particular solution draw a sketch graph of y against tm showing where the graph crosses the t-axis. Determine also the coordinates of the minimum of the point on the sketch graph.

11. $\dfrac{dy}{dx} + y\left(1 + \dfrac{3}{x}\right) = \dfrac{1}{x^2}$, $x > 0$

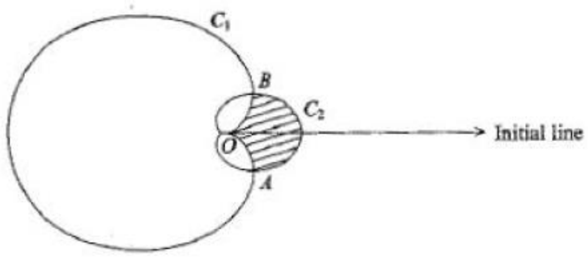
a) Verify that x^3e^x is an integrating factor for the differential equation.

b) Find the general solution of the differential equation.

c) Given that $y = 1$ at $x = 1$, find y at $x = 2$.

12. Find the complete set of values of x for which $\left|x^2 - 2\right| > 2x$.

13.



The figure above is a sketch of two curves C_1 and C_2 with polar equations

$$C_1 : r = 3a(1 - \cos\theta), \qquad -\pi \le \theta < \pi \qquad \text{and}$$
$$C_2 : r = a(1 + \cos\theta), \qquad -\pi \le \theta < \pi$$

The curves meet at the pole O, and at the points A and B.

a) Find, in terms of a, the polar coordinates of the points A and B.

b) Show that the length of the line AB is $\dfrac{3\sqrt{3}}{2}a$.

The region inside C_1 and outside C_2 is shown shaded in the figure above.

c) Find, in terms of a, the area of this region.

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

d) calculate the area of this badge, giving your answer to three significant figures.

14. Given that $y = \tan x$,

a) find $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$

b) Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \dfrac{\pi}{4}\right)$ up to and including the term in $\left(x - \dfrac{\pi}{4}\right)^3$.

c) Hence show that $\tan\dfrac{3\pi}{10} \approx 1 + \dfrac{\pi}{10} + \dfrac{\pi^2}{200} + \dfrac{\pi^3}{3000}$.

15. A complex number z is represented by the point P in the Argand diagram. Given that $|z - 3\mathrm{i}| = 3$,

a) sketch the locus of P.

b) Find the complex number z which satisfies both $|z - 3\mathrm{i}| = 3$ and $\arg(z - 3\mathrm{i}) = \frac{3}{4}\pi$.

The transformation T from the z-plane to the w-plane is given by $w = \dfrac{2\mathrm{i}}{w}$.

c) Show that T maps $|z - 3\mathrm{i}| = 3$ to a line in the w-plane, and give the Cartesian equation of this line.

CHALLENGE QUESTION

Which of the following statements are true and which are false? Justify your answers.

(i) $a^{\ln b} = b^{\ln a}$ for all positive numbers a and b.

(ii) $\cos(\sin\theta) = \sin(\cos\theta)$ for all real θ.

(iii) There exists a polynomial P such that $|\mathrm{P}(x) - \cos x| \leqslant 10^{-6}$ for all (real) x.

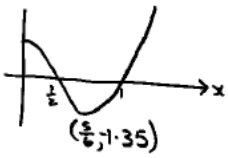
(iv) $x^4 + 3 + x^{-4} \geqslant 5$ for all $x > 0$.

Answers:

Current

1)	2.94 s	2)	-1 ms^{-1}, 0.5 ms^{-1} & 1.75 ms^{-1}	3)	-1.5 ms^{-1}, 0.5 ms^{-1} & 5 ms^{-1}
4a)	$u(1+3e) > 3u \Rightarrow e > \frac{2}{3}$	4b)	*A* moves away from *B*	5)	$\dfrac{5u}{8}$ and $\dfrac{7u}{12}$
6a)	2 m s^{-1}	6b)	12 060 J = 12.06 kJ	7)	0.3 m s^{-1} 3600 J
8)	$v - u$ and $v + \frac{1}{2}u$	9)	$\dfrac{mu}{M+m}$		

Consolidation

10b)	$y = (A + 8t)e^{3t} + 2t^2e^{3t}$	10c)	$y = (3 - 8t + 2t^2)e^{3t}$	10d)	
11b)	$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{c}{x^3}e^{-x}$	11c)	0.171 (3dp) $\frac{1}{6}(1 + e^{-1})$ exact		
12)	$x > 1 + \sqrt{3}$ or $x < -1 + \sqrt{3}$	13a)	$\left(\frac{3a}{2}, \frac{\pi}{3}\right), \left(\frac{3a}{2}, -\frac{\pi}{3}\right)$		
13c)	$a^2[-4\pi + 9\sqrt{3}] \simeq 3.022a^2$	13d)	9.07 cm^2	14a)	$f'(x) = \sec^2 x$ $f''(x) = 2\sec^2 x \tan x$ $f'''(x) = 2\sec^2 x(1 + 3\tan^2 x)$
14b)	$1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$	15a)			
15b)	$z = -\frac{3\sqrt{2}}{2} + \left(3 + \frac{3\sqrt{2}}{2}\right)\mathrm{i}$	15c)	$x = \frac{1}{3}$		

Challenge question: rho

Only integers ending in 1, 3, 7, or 9 are not divisible by 2 or 5. This is 4/10 of the possible integers, so the total number of such integers is 4/10 of 1,000, i.e. 400.

The integers can be added in pairs:

$$\text{sum} = (1+999) + (3+997) + \cdots + (499+501) \, .$$

There are 200 such pairs, so the sum is $1,000 \times 200$ and the average is 500. A simpler argument would be to say that this is obvious by symmetry: there is nothing in the problem that favours an answer greater (or smaller) than 500.

Alternatively, we can find the number of integers divisible by both 2 and 5 by adding the number divisible by 2 (i.e. 500) to the number divisible by 5 (i.e. 200), and subtracting the number divisible by both 2 and 5 (i.e. 100) since these have been counted twice. To find the sum, we can sum those divisible by 2 (using the formula for a geometric progression), add the sum of those divisible by 5 and subtract the sum of those divisible by 10.

For the second part, either of these methods will work. In the first method you consider integers in blocks of 21 (essentially arithmetic to base 21): there are 12 integers in each such block that are not divisible by 3 or 7 (namely 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20) so the total number is $9261 \times 12/21 = 5292$. The average is $9261/2$ as can be seen using the pairing argument $(1+9260)+(2+9259)+\cdots$ or the symmetry argument.

BHASVIC

MαTHS

ASSIGNMENT COVER SHEET rho

Name ______________________________ Maths Teacher ________________________

Question	Done	Backpack	Ready for test	Notes
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