α	β	γ	δ	ε	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o	$\boxed{\pi}$	ρ	σ	τ	υ	φ	χ	ψ	ω

"The mathematician's patterns, like the painter's or the poet's, must be beautiful: the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test." G H Hardy

Further Maths A2 (M2FP2D1) Assignment π (pi) due in 22nd Jan

FP2 PRACTICE PAPER

1. Using the substitution $y = tx$, or otherwise, find the general solution of the differential equation

$$x\frac{dy}{dx} = x + y, \quad x > 0.$$

(7 marks)

2. (*a*) Sketch the curve with equation $y = |x^2 - 8x + 12|$, stating, on your sketch, the coordinates of points at which the curve meets the coordinate axes. **(3 marks)**

(*b*) Hence, or otherwise, find the set of values of x for which $|x^2 - 8x + 12| < 3$. **(7 marks)**

3.

$$u_r = \frac{1}{r+2} - \frac{1}{r+4}$$

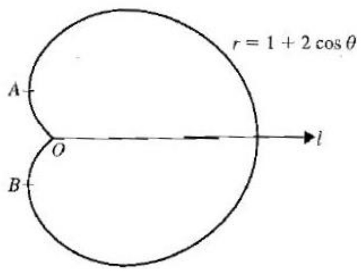
(*a*) Prove that $\sum_{r=1}^{n} u_r = \frac{7}{12} - \frac{2n+7}{(n+3)(n+4)}$. **(5 marks)**

(*b*) Find the exact value of $\sum_{r=6}^{17} u_r$, giving your answer as a single rational number in the form $\frac{p}{q}$, where p and q are integers. **(5 marks)**

4. The polar curve shown has equation $r = 1 + 2\cos\theta$, $-\frac{2}{3}\pi \leqslant \theta \leqslant \frac{2}{3}\pi$. The pole is at O and the initial line is l. At the points A and B on the curve, the tangent to the curve is at right-angles to l. Find

(*a*) the area of the region enclosed by the curve. **(6 marks)**

(*b*) the polar coordinates of A and B. **(6 marks)**



(*a*) One root of the equation $z^3 - 3z - 52 = 0$ is $-2 + 3i$.
(i) Find the other two roots.
(ii) Verify that the sum of all three roots is zero.
(iii) Display the roots on an Argand diagram. **(9 marks)**

(*b*) Find the real numbers p and q for which

$$(5 - 4i)(3 + pi) = q - 2i.$$ **(5 marks)**

PREPARATION *Every week you will be required to do some preparation for future lessons, to be advised by your teacher.*

CURRENT WORK – MECHANICS **Work Energy and Power**

Whenever a numerical value of g is required, take $g = 9.8\,\mathrm{m\,s^{-2}}$.

1. A cyclist and her bicycle have a combined mass of 70 kg. She is cycling at a constant speed of $6\,\mathrm{m\,s^{-1}}$ on a straight road up a hill inclined at 5° to the horizontal. She is working at a constant rate of 480 W. Calculate the magnitude of the resistance to motion from non-gravitational forces.

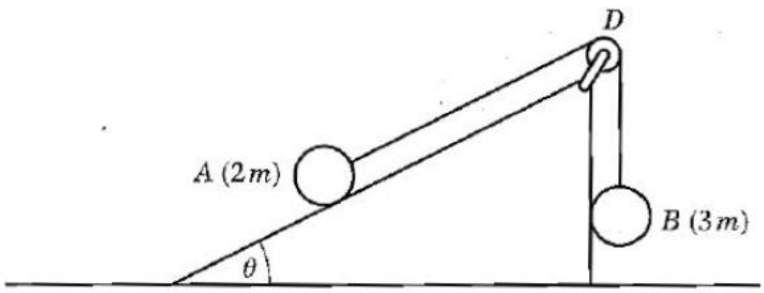
2. A boy hauls a bucket of water through a vertical distance of 25 m. The combined mass of the bucket and water is 12 kg. The bucket starts from rest and finishes at rest.

a Calculate the work done by the boy.

The boy takes 30 s to raise the bucket.

b Calculate the average rate of working of the boy.

3.



The diagram shows a particle A of mass $2m$ which can move on the rough surface of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. A second particle B of mass $3m$ hangs freely attached to a light inextensible string which passes over a smooth pulley fixed at D. The other end of the string is attached to A. The coefficient of friction between A and the plane is $\frac{1}{4}$. The system is released from rest with the string taut and A moves up a line of greatest slope of the plane. When each particle has moved a distance s, A has not reached the pulley and B has not reached the ground.

a Find an expression for the potential energy lost by the system when each particle has moved a distance s.

When each particle has moved a distance s they are moving with speed v.

b Find an expression for v^2, in terms of s.

4. A lorry of mass 16 000 kg is travelling up a straight road inclined at 12° to the horizontal. The lorry is travelling at a constant speed of $14\,\mathrm{m\,s^{-1}}$ and the resistance to motion from non-gravitational forces has a constant magnitude of 200 kN. Find the work done in 10 s by the engine of the lorry.

5. A box of mass 5 kg slides in a straight line across a rough horizontal floor. The initial speed of the box is $10\,\text{m s}^{-1}$. The only resistance to the motion is the frictional force between the box and the floor. The box comes to rest after moving 8 m. Calculate

a the kinetic energy lost by the box in coming to rest,

b the coefficient of friction between the box and the floor.

6. A car of mass 900 kg is moving along a straight horizontal road. The resistance to motion has a constant magnitude. The engine of the car is working at a rate of 15 kW. When the car is moving with speed $20\,\text{m s}^{-1}$, the acceleration of the car is $0.3\,\text{m s}^{-2}$.

a Find the magnitude of the resistance.

The car now moves downhill on a straight road inclined at 4° to the horizontal. The engine of the car is now working at a rate of 8 kW. The resistance to motion from non-gravitational forces remains unchanged.

b Calculate the speed of the car when its acceleration is $0.5\,\text{m s}^{-2}$.

7. A block of wood of mass 4 kg is pulled across a rough horizontal floor by a rope inclined at 15° to the horizontal. The tension in the rope is constant and has magnitude 75 N. The coefficient of friction between the block and the floor is $\frac{3}{8}$.

a Find the magnitude of the frictional force opposing the motion.

b Find the work done by the tension when the block moves 6 m.

The block is initially at rest.

c Find the speed of the block when it has moved 6 m.

8. The engine of a lorry works at a constant rate of 20 kW. The lorry has a mass of 1800 kg. When moving along a straight horizontal road there is a constant resistance to motion of magnitude 600 N. Calculate

a the maximum speed of the lorry,

b the acceleration of the lorry, in m s^{-2}, when its speed is $20\,\text{m s}^{-1}$.

9. A car of mass 1200 kg is travelling at a constant speed of $20\,\text{m s}^{-1}$ along a straight horizontal road. The constant resistance to motion has magnitude 600 N.

a Calculate the power, in kW, developed by the engine of the car.

The rate of working of the engine of the car is suddenly increased and the initial acceleration of the car is $0.5\,\text{m s}^{-2}$. The resistance to motion is unchanged.

b Find the new rate of working of the engine of the car.

The car now comes to a hill. The road is still straight but is now inclined at 20° to the horizontal. The rate of working of the engine of the car is increased further to 50 kW. The resistance to motion from non-gravitational forces still has magnitude 600 N. The car climbs the hill at a constant speed $V\,\text{m s}^{-1}$.

c Find the value of V.

CONSOLIDATION – FP2

10. a) Find the value of λ for which $\lambda x\cos 3x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+9y=-12\sin 3x \qquad \text{(I)}$$

b) Hence find the general solution of this differential equation.

The particular solution of the differential equation for which y = 1 and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ at x = 0, is y = g(x).

c) Find g(x).

d) Sketch the graph of $y = \mathrm{g}(x), 0 \le x \le \pi$.

11. a) Using the substitution $t = x^2$, or otherwise, find $\int x^3 e^{-x^2} \mathrm{d}x$.

b) Find the general solution of the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + 3 = xe^{-x^2}$, $x > 0$.

12. a) Sketch, on the same axes, the graph of $y = |(x-2)(x-4)|$, and the line with equation $y = 6 - 2x$.

b) Find the exact values of x for which $|(x-2)(x-4)| = 6 - 2x$.

c) Hence solve the inequality $|(x-2)(x-4)| < 6 - 2x$.

13. a) Sketch the curve with polar equation $r = 3\cos 2\theta$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

b) Find the area of the smaller finite region enclosed between the curve and the half-line $\theta = \frac{\pi}{6}$.

c) Find the exact distance between the two tangents which are parallel to the initial line.

14. The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}} , \quad z \ne -\mathrm{i} .$$

a) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane.

b) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane.

c) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane.

d) Mark on your sketches the point P, where z = I, and its image Q under T in the w-plane.

CHALLENGE QUESTION

Show, by means of a change of variable or otherwise, that

$$\int_0^\infty \mathrm{f}\left((x^2+1)^{1/2}+x\right)\mathrm{d}x = \frac{1}{2}\int_1^\infty (1+t^{-2})\mathrm{f}(t)\,\mathrm{d}t\ ,$$

for any given function f .

Hence, or otherwise, show that

$$\int_0^\infty \left((x^2+1)^{1/2}+x\right)^{-3}\mathrm{d}x = \frac{3}{8}\,.$$

Answers:

FP2 Practice Paper:

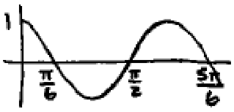
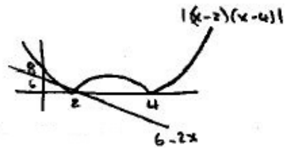
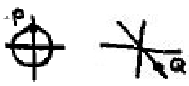
1)	$y = x\ln(kx)$	2b)	$4-\sqrt{7}<x<3$ or $5<x<4+\sqrt{7}$	3b)	$\frac{349}{2520}$
4a)	$2\pi+\frac{3\sqrt{3}}{2}$	4b)	$(\frac{1}{2},1.82),(\frac{1}{2},-1.82)$	5a)	**i)** 4, −2−3i

5b) p = 2, *q* = 23

Current:

1)	20.2 N	2a)	2940 J	2b)	98 J s^{-1} (or 98W)
3a)	$\frac{9mgs}{5}$	3b)	$\frac{14gs}{25}$	4)	32 600 000 J (or 32 600 kJ)
5a)	250 J	5b)	0.638 m s^{-1}	6a)	480 N
6b)	25.4 m s^{-1}	7a)	7.42 N	7b)	435 J
7c)	14.0 m s^{-1}	8a)	33.3 m s^{-1}	8b)	0.222 m s^{-2}
9a)	12 kW	9b)	24 kW	9c)	10.8

Consolidation:

10a)	$\lambda = 2$	10b)	$y = A\sin 3x + B\cos 3x + 2x\cos 3x$	10c)	$g(x) = \cos 3x + 2x\cos 3x$
10d)		11a)	$-\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + c$	11b)	$x^3y = -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + c$
12a)		12b)	$2-\sqrt{2},\ 4-\sqrt{2}$	12c)	$2-\sqrt{2} < x < 4-\sqrt{2}$
13a)		13b)	$\frac{9}{2}\left[\frac{\pi}{24}-\frac{\sqrt{3}}{16}\right] \cong 0.103$	13c)	$\frac{2}{3}\sqrt{6}$
14b)	Line $y=-x$	14c)		14d)	See previous diagrams

Past Challenge solutions

Challenge Question – mu:

We will write t for $\tan\theta$ (or $\tan\phi$) throughout.

First part : $\tan 2\theta = \dfrac{2t}{1-t^2} = -1$ (since $t^2 = 2t+1$ is given).

For the second part, we first work out $\tan 3\theta$. We have

$$\tan 3\theta = \tan(\theta + 2\theta) = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \frac{t + 2t/(1-t^2)}{1 - t(2t/(1-t^2))} = \frac{3t - t^3}{1 - 3t^2},$$

so the equation becomes

$$\frac{3t - t^3}{1 - 3t^2} = t - 2, \quad \text{i.e.} \quad t^3 - 3t^2 + t + 1 = 0.$$

One solution (by inspection) is $t = 1$. Thus one set of roots is given by $\theta = n\pi + \pi/4$.

There are no other obvious integer roots, but we can reduce the cubic equation to a quadratic equation by dividing out the known factor $(t-1)$. I would write $t^3 - 3t^2 + t + 1 = (t-1)(t^2 + at - 1)$ since the coefficients of t^2 and of t^0 in the quadratic bracket are obvious. Then I would multiply out the brackets to find that $a = -2$. Now we see the connection with the first part: $t^2 + at - 1 = 0 \Rightarrow \tan 2\theta = -1$, and hence $2\theta = n\pi - \pi/4$.

The roots are therefore $\theta = n\pi + \pi/4$ and $\theta = n\pi/2 - \pi/8$. The multiples of $\pi/8$ in the given range are $\{2, 3, 7, 10, 11, 15\}$.

For the last part, we could set $\cot\phi = 1/\tan\phi$ and $\cot 3\phi = 1/\tan 3\phi$ thereby obtaining

$$\frac{1}{t} = 2 + \frac{1 - 3t^2}{3t - t^3}.$$

This simplifies to the cubic equation $t^3 + t^2 - 3t + 1 = 0$. There is an integer root $t = 1$, and the remaining quadratic is $t^2 + 2t - 1 = 0$. Learning from the first part, we write this as $\dfrac{2t}{1-t^2} = 1$, which means that $\tan 2\phi = n\pi + \pi/4$. Proceeding as before gives (noting the different range) the following multiples of $\pi/8$: $\{2, 1, -3, -6, -7, -11\}$.

Challenge Question – nu:

Clearly, to get the argument of f right, we must set

$$t = (x^2+1)^{1/2} + x\,.$$

We must check that the new limits are correct (if not, we are completely stuck). When $x = 0$, $t = 1$ as required. Also, $(x^2+1)^{1/2} \to \infty$ as $x \to \infty$, so $t \to \infty$. Thus the upper limit is still ∞, again as required.

The transformed integral is

$$\int_1^\infty \mathrm{f}\,(t)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

so the next task is to find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$. This we can do in two ways: we find x in terms of t and differentiate it; or we could find $\dfrac{\mathrm{d}t}{\mathrm{d}x}$ and turn it upside down. The snag with the second method is that the answer will be in terms of x, so we will have to express x in terms of t anyway — in which case, we may as well use the first method.

We start by finding x in terms of t:

$$t = (x^2+1)^{1/2} + x \Rightarrow (t-x)^2 = (x^2+1) \Rightarrow x = \frac{t^2-1}{2t} = \frac{t}{2} - \frac{1}{2t}\,.$$

Then we differentiate:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2} + \frac{1}{2t^2} = \frac{1}{2}(1+t^{-2})$$

which agrees exactly the factor that appears in the second integrand.

For the last part, we take $\mathrm{f}\,(t) = t^{-3}$. Thus

$$\begin{aligned}\int_0^\infty \left((x^2+1)^{1/2}+x\right)^{-3}\mathrm{d}x &= \frac{1}{2}\int_1^\infty (t^{-3}+t^{-5})\,\mathrm{d}t\\ &= \frac{1}{2}\left(\frac{t^{-2}}{-2}+\frac{t^{-4}}{-4}\right)\Bigg|_1^\infty\\ &= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3}{8}\,.\end{aligned}$$

as required.

Challenge Question: xi

(i) True. The easiest way to see this is to log both sides. For the left hand side, we have

$$\ln(a^{\ln b}) = (\ln b)(\ln a)$$

and for the right hand side we have

$$\ln(a^{\ln b}) = (\ln a)(\ln b)\,,$$

which agree.

Note that we have to be a bit careful with this sort of argument. The argument used is that $A = B$ because $\ln A = \ln B$. This requires the property of the ln function that $\ln A = \ln B \Rightarrow A = B$. You can easily see that this property holds because ln is a strictly increasing function; if $A > B$, then $\ln A > \ln B$. The same would not hold for (say) sin (i.e. $\sin A = \sin B \not\Rightarrow A = B$).

(ii) False. $\theta = \pi/2$ is an easy counterexample.

(iii) False. Roughly speaking, any polynomial can be made as large as you like by taking x to be very large (provided it is of degree greater than zero), whereas $|\cos x| \leqslant 1$. There is obviously no polynomial of degree zero (i.e. no constant number) for which the statement holds.

(iv) True: $x^4 + 3 + x^{-4} = (x^2 - x^{-2})^2 + 5 \geqslant 5$.

Challenge Question: omicron

Let the circle have diameter d and let the length of one side of the rectangle be x and the length of the adjacent side be y.

Then, by Pythagoras's theorem,

$$y = \sqrt{d^2 - x^2} \qquad (\dagger)$$

and the perimeter P is given by

$$P = 2x + 2\sqrt{d^2 - x^2}\ .$$

We can find the largest possible value of P as x varies by calculus. We have

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 2 - 2\frac{x}{\sqrt{d^2 - x^2}}\ ,$$

so for a stationary point, we require (cancelling the factor of 2 and squaring)

$$1 = \frac{x^2}{d^2 - x^2} \qquad \text{i.e.} \qquad 2x^2 = d^2\ .$$

Thus $x = d/\sqrt{2}$ (ignoring the negative root for obvious reasons). Substituting this into ($\dagger$) gives $y = d/\sqrt{2}$, so the rectangle is indeed a square, with perimeter $2\sqrt{2}\,d$.

But is it the maximum perimeter? The easiest way to investigate is to calculate the second derivative, which is easily seen to be negative for all values of x, and in particular when $x = d/\sqrt{2}$. The stationary point is certainly a maximum.

For the second part, we consider

$$\mathrm{f}\,(x) = x^n + (d^2 - x^2)^{n/2}\,.$$

The first thing to notice is that f is constant if $n = 2$, so in this case, the largest (and smallest) value is d^2. For $n = 3$, we have

$$\mathrm{f}'(x) = 3x^2 - 3x(d^2 - x^2)^{1/2},$$

so $\mathrm{f}\,(x)$ is stationary when $x^4 = x^2(d^2 - x^2)$, i.e. when $2x^2 = d^2$ as before or when $x = 0$. The corresponding stationary values are $\sqrt{2}d^3$ and $2d^3$, so this time the largest value occurs when $x = 0$.

Challenge Question: pi

First we expand binomially:

$$\begin{aligned}
&\left(1-\frac{2}{100}\right)^{\frac{1}{2}}\\
\approx\ & 1+\left(\frac{1}{2}\right)\left(-\frac{2}{100}\right)+\left(\frac{1}{2!}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{2}{100}\right)^2-\left(\frac{1}{3!}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{2}{100}\right)^3+\cdots\\
=\ & 1-\frac{1}{100}-\frac{0.5}{10^4}-\frac{0.5}{10^6}=0.9899495\,.
\end{aligned}$$

It is clear that the next term in the expansion would introduce the seventh and eighth places of decimals, which it seems we do not need. Of course, after further manipulations we might find that the above calculation does not supply the 6 decimal places we need, in which case we will work out the next term in the expansion.

But $\left(\dfrac{98}{100}\right)^{\frac{1}{2}}=\dfrac{7\sqrt{2}}{10}$, so $\sqrt{2}\approx 9.899495/7\approx 1.414214\,.$

Second part:

$$\begin{aligned}
&\left(1+\frac{3}{125}\right)^{\frac{1}{3}}\\
\approx\ & 1+\left(\frac{1}{3}\right)\left(\frac{3}{125}\right)+\left(\frac{1}{2!}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{3}{125}\right)^2+\left(\frac{1}{3!}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{3}{125}\right)^3+\cdots\\
=\ & 1-\frac{8}{1000}-\frac{64}{10^6}+\frac{5}{3}\frac{8^3}{10^9}\\
=\ & 1.007936+\frac{256}{3}\frac{1}{10^8}=1.007937\,.
\end{aligned}$$

Successive terms in the expansion decrease by a factor of about 1000, so this should give the right number of decimal places.

But $\left(\dfrac{128}{125}\right)^{\frac{1}{3}}=\dfrac{4\sqrt[3]{2}}{5}=\dfrac{8\sqrt[3]{2}}{10}$ so $\sqrt[3]{2}\approx 10.0793/8\approx 1.259921\,.$

BHASVIC

MαTHS

ASSIGNMENT COVER SHEET pi

Name ______________________ Maths Teacher ______________________

Question	Done	Backpack	Ready for test	Notes
FP2 Practice Paper				
1				
2				
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